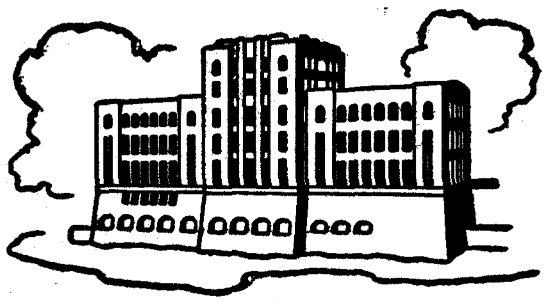
# ON THE EQUATIONS OF A THICK AXISYMMETRIC TURBULENT BOUNDARY LAYER

by

V. C. Patel



Sponsored by
General Hydromechanics Research Program
of the Naval Ship Systems Command
Naval Ship Research and Development Center
Contract No. N00014-68-A-0196-0002



IIHR Report No. 143

Iowa Institute of Hydraulic Research
The University of Iowa
Iowa City, Iowa

January 1973

Approved for public release; distribution unlimited

# ON THE EQUATIONS OF A THICK AXISYMMETRIC TURBULENT BOUNDARY LAYER

by

V. C. Patel

Sponsored by
General Hydromechanics Research Program
of the Naval Ship Systems Command
Naval Ship Research and Development Center
Contract No. N00014-68-A-0196-0002

IIHR Report No. 143

Iowa Institute of Hydraulic Research
The University of Iowa
Iowa City, Iowa

January 1973

## ABSTRACT

From an examination of the Reynolds equations for axisymmetric turbulent flow in situations where the thickness of the boundary layer is of the same order as the transverse radius of curvature of the surface, it has been shown that, in general, neither the boundary layer nor the potential flow outside it can be calculated independently of the other, owing to significant interaction between the two flow regimes. Following a discussion of the various procedures for extending conventional thin boundary-layer calculation methods to treat thick axisymmetric turbulent boundary-layers, taking into account the influence of transverse curvature either at the differential or the integral level, a method is proposed for the simultaneous solution of the boundary layer and the potential flow equations, allowing the two flow regimes to interact.

## ACKNOWLEDGEMENT

This report is based upon research conducted under the General Hydromechanics Research Program of the Naval Ship Systems Command, technically administered by the Naval Ship Research and Development Center, under Contract N00014-68-A-0196-0002.

## LIST OF CONTENTS



## LIST OF FIGURES

# ON THE EQUATIONS OF A THICK AXISYMMETRIC TURBULENT BOUNDARY LAYER

## I. INTRODUCTION

This paper is concerned with the equations of an incompressible turbulent boundary layer developing on a body of revolution. As evidenced by the Stanford Conference (Kline *et al.* 1968), nominally two-dimensional turbulent boundary layers developing on surfaces of small curvature have been studied extensively, both experimentally and theoretically. Although the axisymmetric case is of considerable interest in many engineering applications it does not appear to have received any special attention in the past. The reason for ignoring this aspect almost certainly lies in the often quoted observation that the influence of transverse curvature can be neglected provided the boundary layer thickness is small in comparison with the local radius of the body. This of course begs the question: How small does the boundary layer thickness need to be for this approximation to hold? If one considers the boundary layer on a long thin cylinder of constant radius, with its axis oriented along the flow, it is clear that this approximation may not be applicable sufficiently far from the leading edge. Furthermore, on a body of revolution of finite length there is always a region close to the tail where the boundary layer thickness may become much larger than the local radius of the body. The purpose of this paper is to explore the equations of such thick axisymmetric turbulent boundary layers and examine the possibilities of extending conventional thin boundary-layer calculation procedures to the solution of these equations.

## II. DIFFERENTIAL EQUATIONS OF THICK AXISYMMETRIC BOUNDARY LAYERS

Let us consider axially symmetric flow on a body of revolution whose transverse and longitudinal radii of curvature are $r_o$ and $R$ ($\equiv 1/\kappa$), respectively. We choose a curvilinear co-ordinate system (x,y,z), where x is the distance measured along a meridian, y is the distance measured normal to the surface of the body, and z is the azimuthal angle. Then, from geometry (see Figure 1) we have

$$\left.\begin{aligned} r &= r_o + y\cos\phi, \\ \frac{dr_o}{dx} &= \sin\phi, \\ \frac{d\phi}{dx} &= -1/R = -\kappa, \\ \frac{\partial r}{\partial y} &= \cos\phi, \\ \frac{\partial r}{\partial x} &= (1+\frac{y}{R})\sin\phi = (1+\kappa y)\sin\phi, \end{aligned}\right\} \quad (1)$$

where r is the distance from the axis of the body and $\phi$ is the angle between the axis and the tangent to the meridian. The metric coefficients, or linearizing factors, associated with the x, y, z, directions are

$$h_1 = 1+\kappa y, \quad h_2 = 1, \quad h_3 = r, \qquad (2)$$

respectively.

Following Nash and Patel (1972), the equations of conservation of mean-flow momentum, i.e. the Reynolds equations, for steady turbulent flow in the above co-ordinate system may then be written as follows:

$$\begin{aligned} &\frac{U}{h_1}\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\kappa}{h_1}UV + \frac{1}{h_1}\frac{\partial}{\partial x}(\frac{p}{\rho} + \overline{u^2}) + \frac{\partial}{\partial y}(\overline{uv}) \\ &+ (\frac{2\kappa}{h_1} + \frac{\cos\phi}{r})\overline{uv} + \frac{\sin\phi}{r}\overline{u^2} - \frac{\sin\phi}{r}\overline{w^2} \\ &- \nu\Big[\frac{1}{h_1^2}\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - (\frac{y}{h_1^2}\frac{d\kappa}{dx} - \frac{\sin\phi}{r})\frac{1}{h_1}\frac{\partial U}{\partial x} \\ &+ (\frac{\kappa}{h_1} + \frac{\cos\phi}{r})\frac{\partial U}{\partial y} + \frac{2\kappa}{h_1^2}\frac{\partial V}{\partial x} - (\frac{\kappa^2}{h_1^2} + \frac{\sin^2\phi}{r^2})U \\ &+ (\frac{1}{h_1^2}\frac{d\kappa}{dx} - \frac{\kappa y}{h_1^3}\frac{d\kappa}{dx} - \frac{\sin 2\phi}{2r^2} + \frac{\sin\phi}{r}\frac{\kappa}{h_1})V\Big] = 0, \qquad (3) \end{aligned}$$

$$\frac{U}{h_1}\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} - \frac{\kappa}{h_1}U^2 + \frac{1}{h_1}\frac{\partial}{\partial x}(\overline{uv}) + \frac{\partial}{\partial y}\left(\frac{p}{\rho} + \overline{v^2}\right)$$

$$+ \frac{\sin\phi}{r}\overline{uv} - \frac{\kappa}{h_1}\overline{u^2} + \left(\frac{\cos\phi}{r} + \frac{\kappa}{h_1}\right)\overline{v^2} - \frac{\cos\phi}{r}\overline{w^2}$$

$$- \nu\left[\frac{1}{h_1^2}\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - \left(\frac{y}{h_1^2}\frac{d\kappa}{dx} - \frac{\sin\phi}{r}\right)\frac{1}{h_1}\frac{\partial V}{\partial x}\right.$$

$$+ \left(\frac{\kappa}{h_1} + \frac{\cos\phi}{r}\right)\frac{\partial V}{\partial y} - \frac{2\kappa}{h_1^2}\frac{\partial U}{\partial x} - \left(\frac{\cos^2\phi}{r^2} + \frac{\kappa^2}{h_1^2}\right)V$$

$$\left. + \left(-\frac{1}{h_1^2}\frac{d\kappa}{dx} + \frac{\kappa y}{h_1^3}\frac{d\kappa}{dx} - \frac{\sin 2\phi}{2r^2} - \frac{\sin\phi}{r}\frac{\kappa}{h_1}\right)U\right] = 0. \tag{4}$$

The equation of continuity in this co-ordinate system becomes

$$\frac{1}{h_1}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\sin\phi}{r}U + \left(\frac{\cos\phi}{r} + \frac{\kappa}{h_1}\right)V = 0. \tag{5}$$

Here, U and V are the components of mean velocity in directions x and y, respectively. The velocity fluctuations in the x, y, z directions are denoted by u, v, w, respectively. p is static pressure, $\rho$ is density and $\nu$ is kinematic viscosity.

In order to treat boundary layer flow on a body of revolution we recognize that, in general, the body has three distinct length scales, namely the overall length L, the longitudinal radius of curvature R, and the transverse radius $r_o$. In the usual thin boundary-layer theory it is assumed that the thickness of the layer, $\delta$, is everywhere at least an order of magnitude smaller than the three length scales, i.e. $\delta/L \ll 1$. $\delta/R \ll 1$ and $\delta/r_o \ll 1$. With this assumption, order of magnitude considerations applied to equations (3), (4) and (5), lead to the well known thin axisymmetric boundary-layer equations:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \frac{\partial}{\partial y}(\overline{uv}) - \nu\frac{\partial^2 U}{\partial y^2} = 0, \tag{6}$$

$$\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \overline{v^2}\right) = 0, \tag{7}$$

and

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + U \frac{1}{r_o} \frac{dr_o}{dx} = 0. \qquad (8)$$

These are identical with the equations for plane surface flows except for the additional transverse curvature term in the equation of continuity. This additional term presents no difficulty and many of the existing integral as well as differential calculation methods, constructed for plane surface boundary layers, can readily be extended to calculate the development of a thin boundary layer on a body of revolution. In what follows, however, we shall consider the situation in which the boundary layer may be regarded as thin in comparison with the overall length L and the longitudinal radius of curvature R but not in comparison with the local transverse curvature of the body. We shall therefore examine situations in which $\delta/L \ll 1$, $\delta/R \ll 1$ and $\delta/r_o \sim 1$. Two flows of practical interest where these conditions are closely realized are sketched in Figure 2. In both cases we have boundary layer behavior since, according to the rather broad definition, the direct influence of viscosity and turbulence is still confined to a "narrow" region (in comparison with the infinite expanse of the overall flow field) close to the boundaries. Although the two cases appear to have a number of features in common we shall find later that they behave in quite different fashions and consequently have to be treated in quite different ways. The analysis of both, however, starts by examining the Reynolds and continuity equations with the longitudinal curvature terms neglected, viz

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{\partial}{\partial x} \left(\frac{p}{\rho} + \overline{u^2}\right) + \frac{1}{r} \frac{\partial}{\partial y} (r\overline{uv}) + \frac{\sin\phi}{r} (\overline{u^2} - \overline{w^2})$$

$$- \nu \left[\frac{1}{r} \frac{\partial}{\partial x} \left(r \frac{\partial U}{\partial x}\right) + \frac{1}{r} \frac{\partial}{\partial y} \left(r \frac{\partial U}{\partial y}\right) - \left(\frac{\sin\phi}{r}\right)^2 U - \frac{\sin 2\phi}{2r^2} V\right] = 0, \qquad (9)$$

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + \frac{\partial}{\partial y} \left(\frac{p}{\rho} + \overline{v^2}\right) + \frac{1}{r} \frac{\partial}{\partial x} (r\overline{uv}) + \frac{\cos\phi}{r} (\overline{v^2} - \overline{w^2})$$

$$- \nu \left[\frac{1}{r} \frac{\partial}{\partial x} \left(r \frac{\partial V}{\partial x}\right) + \frac{1}{r} \frac{\partial}{\partial y} \left(r \frac{\partial V}{\partial y}\right) - \frac{\sin 2\phi}{2r^2} U - \left(\frac{\cos\phi}{r}\right)^2 V\right] = 0, \qquad (10)$$

$$\frac{\partial}{\partial x} (Ur) + \frac{\partial}{\partial y} (Vr) = 0. \qquad (11)$$

We now proceed to see what simplifications, if any, can be made in these equations for the two cases shown in Figure 2.

We consider first the case of a thick boundary layer growing on a long slender cylinder of constant radius, so that $\phi = 0$. Here, it is reasonable to assume that, regardless of the relative thickness of the boundary layer, the mean-flow streamlines remain nearly parallel to the surface so that the normal component of mean velocity is much smaller than the longitudinal component, i.e. $V \ll U$. We further assume that the Reynolds stresses will be similar in magnitude to those occurring in a flat-plate boundary layer. With these assumptions, order of magnitude considerations applied to equations (9),(10), and(11) lead to

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{1}{r} \frac{\partial}{\partial y} \{r \; (\overline{uv} - \nu\frac{\partial U}{\partial y})\} = 0 \tag{12}$$

and

$$r \frac{\partial U}{\partial x} + \frac{\partial}{\partial y} (Vr) = 0. \tag{13}$$

Notice that, within the approximations made, the boundary layer on a cylinder of constant radius develops in a constant pressure field. Pressure variations across the boundary layer do, however, occur not only as a result of the normal Reynolds stresses but also due to the displacement effect of the boundary layer. In other words, the external potential flow behaves as if it were developing on a cylinder of ever-increasing radius. Detailed calculations of boundary layer growth, however, indicate that although the boundary layer itself may be very thick in comparison with the radius of the cylinder the rate of increase of the displacement thickness is small, and comparable with that occurring on a flat plate. This implies that the self-induced pressure gradients, and consequently the curvatures of the mean-flow streamlines, are negligible. In the present case we may therefore say that the INTERACTION between the boundary layer and the external potential flow is WEAK and the equations of the boundary layer, namely equations (12) and (13), can be solved once the velocity of the external flow is specified.

Equations (12) and (13) have indeed been used by a number of workers to study the influence of transverse surface curvature on the development of laminar as well as turbulent boundary layers. Cebeci (1970) has recently reviewed the previous studies and also presented results obtained by the solution of equations (12) and (13) using finite-difference techniques. For laminar flow the numerical results of Cebeci showed excellent agreement with the previous analytic studies of Seban and Bond (1951), Kelly (1954), and Stewartson (1955). In the case of turbulent flow Cebeci employed an eddy-viscosity model for the Reynolds shear stress, with the additional assumption that this model is not directly affected by transverse curvature, and obtained good agreement with the experimental data of Richmond (1957) and Yasuhara (1959), collected from turbulent boundary layers on slender cylinders of constant radius. The overall success of Cebeci's calculations would appear to vindicate the assumptions made in deducing equations (12) and (13).

We consider next the second case shown in Figure 2, namely the boundary layer near the conical tail of a body of revolution. Here, considerations of the principle of conservation of mass, applied to the flow within the boundary layer, immediately lead to the conclusion that the diminishing radius of the body must be accompanied by a rapid thickening of the boundary layer. This thickening is associated not so much with the adverse longitudinal pressure-gradients which are undoubtedly present as with the changing geometry of the surface. In the absence of premature separation prior to this thickening we have a situation in which the boundary layer thickness may be much larger than the local radius of the body. While this suggests that transverse curvature effects will be present here just as in the previous case, there is an important difference in that the rapid thickening of the boundary layer leads to appreciable divergence of the mean-flow streamlines in planes normal to the surface and consequently it is no longer possible to assume $V \ll U$. In other words, we now have substantial variation of pressure across the boundary layer. This variation of pressure is of course accounted for by the y-momentum equation, equation (10). The flow close to the tail of a body of revolution is thus characterized by a STRONG INTERACTION between the boundary layer and the external potential flow, with the result that neither can be determined independently of the other. In particular, we can no longer use potential flow pressure distribution on the wall, together with the usual constant pressure assumption (for the y-direction), to calculate the boundary layer development. Special iterative techniques in which potential flow and boundary layer calculations are performed simultaneously have to be devised. We shall discuss these later on in the paper.

Returning to the general Reynolds equations, i.e. equations (9), (10) and (11), we see that the only simplification that can be made in these is that we can neglect some of the viscosity terms in the momentum equations since they are small and important only in the sublayer. In the absence of any prior knowledge of the importance of the turbulence terms we shall retain them in the analysis for the present. If equation (10), with the viscous terms neglected, is integrated with respect to y, making some use of the equation of continuity, there results

$$\frac{p_w - p}{\rho} = V^2 + \overline{v^2} + \frac{\partial}{\partial x} \int_0^y (UV + \overline{uv})\, dy + \sin\phi \int_0^y \frac{UV + \overline{uv}}{r}\, dy + \cos\phi \int_0^y \frac{V^2 + \overline{v^2} - \overline{w^2}}{r}\, dy, \qquad (14)$$

where $p_w(x)$ is the pressure distribution on the wall, y=0. The recent experiments of Patel, Nakayama and Damian (1973) indicate that the Reynolds stresses are much smaller than the corresponding products of mean velocity components when the boundary layer is thick, i.e. $\overline{uv} \ll UV$ and $\overline{v^2} \ll V^2$. This leads to the conclusion that the static pressure variation across the thick axisymmetric boundary layer is associated primarily with the mean flow. Consequently, the differential equations for this case may be written

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial y} \{r(\overline{uv} - \nu \frac{\partial U}{\partial y})\} = 0, \qquad (15)$$

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0, \qquad (16)$$

$$\frac{\partial}{\partial x} (Ur) + \frac{\partial}{\partial y} (Vr) = 0. \qquad (17)$$

These equations form an elliptic set since they differ from Euler's potential flow equations only in the presence of the viscous and Reynolds shear-stress terms in the first equation.

## III. CLOSURE OF THE DIFFERENTIAL EQUATIONS.

It will be noticed that equations (15) through (17) include the somewhat simpler equations of the boundary layer on a cylinder of constant radius as a special case. We shall therefore confine the subsequent discussion to the general case. These equations contain four unknowns, namely U, V, p and $\overline{uv}$. In order to make them determinate it is therefore necessary to furnish one additional equation. This usually implies some assumption concerning the Reynolds stress $\overline{uv}$.

Perhaps the simplest way to effect closure of the differential equations is to employ the classical phenomenological theories in which the Reynolds stress is related to the mean flow via mixing-length or eddy-viscosity functions. This, however, raises an additional uncertainty concerning the particular variation of mixing length or eddy viscosity. through the boundary layer that has to be chosen. In the treatment of the thick boundary layer on a cylinder Cebeci (1970) made the assumption that the eddy-viscosity model found most suitable for thin, plane-surface boundary layers applies equally well to thick axisymmetric boundary layers. This implies that there is no direct influence of transverse curvature on the eddy viscosity. Bradshaw (1969) and others, on the other hand, suggest that longitudinal surface curvature has a marked influence on the turbulence structure, and by implication, on the mixing-length and eddy-viscosity distributions through the boundary layer. The recent experiments of Patel, Nakayama and Damian (1973) in the thick boundary layer near the tail of a body of revolution also showed that mixing length as well as eddy viscosity are influenced directly and significantly by transverse curvature. The problem of determining this influence quantitatively is therefore an important one and worthy of further research.

The closure of the differential equations can also be effected by a number of other methods. Here we shall single out for discussion the procedure associated with the names of Townsend, Rotta, Bradshaw and others, in which a rate equation for the Reynolds stress $\overline{uv}$ is obtained by postulating plausible models for the various terms in the turbulent kinetic-energy equation. For a thin boundary layer developing on a plane surface the turbulent kinetic-energy equation may be written

$$\underbrace{U \frac{\partial}{\partial x}(\overline{q^2}/2) + V \frac{\partial}{\partial y}(\overline{q^2}/2)}_{\text{convection}} + \underbrace{\frac{\partial U}{\partial y} \overline{uv}}_{\text{production}} + \underbrace{\frac{\partial}{\partial y} (\overline{\frac{vq^2}{2} + \frac{vp'}{\rho}})}_{\text{diffusion}} + \underbrace{\varepsilon}_{\text{dissipation}} = 0, \qquad (18)$$

where $\frac{1}{2}\rho q^2 \equiv \frac{1}{2}\rho(\overline{u^2}+\overline{v^2}+\overline{w^2})$ is turbulent kinetic-energy and $\varepsilon$ represents dissipation into heat. Following Townsend and Rotta, Bradshaw, Ferriss and Atwell (1967) assume that

$$-\frac{\overline{uv}}{\overline{q^2}} = a_1, \quad (\overline{\frac{vq^2}{2} + \frac{vp'}{\rho}}) = G\overline{q^2}(\overline{q^2})_{max}^{1/2}, \quad \varepsilon = \frac{(\overline{q^2})^{3/2}}{L_D}, \qquad (19)$$

where $a_1$ is a constant (=0.15), while the diffusion function G and the dissipation length $L_D$ are functions of $y/\delta$ only. Introduction of equations (19) in equation (18) leads to the required rate equation for the Reynolds stress, $\tau \equiv -\rho\overline{uv}$ , in the form

$$\frac{1}{2a_1} \{U \frac{\partial\tau}{\partial x} + V \frac{\partial\tau}{\partial y}\} - \tau \frac{\partial U}{\partial y} + \frac{\partial}{\partial y} \{G \frac{\tau}{a_1^{3/2}} \sqrt{\frac{\tau_{max}}{\rho}}\} + \frac{1}{a_1^{3/2} L_D} \frac{\tau^{3/2}}{\rho^{1/2}} = 0. \qquad (20)$$

It can be shown that, when the convection and diffusion terms in this equation are neglected, the resulting equation (Production = Dissipation) reduces to the familiar mixing length formula

$$\tau = \rho L^2 (\frac{\partial U}{\partial y})^2 ,$$

the mixing-length L being related to the dissipation length $L_D$ by

$$L = a_1^{3/2} L_D.$$

For the thick axisymmetric boundary layer the turbulent kinetic-energy equation, correct to the approximations already introduced, may be written

$$\underbrace{U \frac{\partial}{\partial x}(\overline{q^2}/2) + V \frac{\partial}{\partial y}(\overline{q^2}/2)}_{\text{convection}} + \underbrace{\frac{\partial U}{\partial x} \overline{u^2} + \frac{\partial V}{\partial y} \overline{v^2} + (\frac{\sin\phi}{r} U + \frac{\cos\phi}{r} V)\overline{w^2} + (\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}) \overline{uv}}_{\text{production}}$$

$$+ \underbrace{\frac{1}{r} \frac{\partial}{\partial x} \{r (\overline{\frac{uq^2}{2} + \frac{up'}{\rho}})\} + \frac{1}{r} \frac{\partial}{\partial y} \{r (\overline{\frac{vq^2}{2} + \frac{vp'}{\rho}})\}}_{\text{diffusion}} + \underbrace{\varepsilon}_{\text{dissipation}} = 0. \qquad (21)$$

In order to develop a rate equation similar to equation (20), retaining all the essential elements of the method of Bradshaw *et al.*, it is necessary not only to assess the effect of the extra production and diffusion terms on the overall energy balance but also to say something about the direct influence of transverse curvature on the three empirical functions $a_1$, G and $L_D$ (or L). Of the production terms, the dominant one is $\frac{\partial U}{\partial y}\overline{uv}$, as is the case in thin boundary layers. Since V is no longer small, however, the other terms, although smaller than the dominant one, cannot be neglected a priori. Fortunately the turbulence measurements of Patel, Nakayama and Damian cited earlier indicate that in a thick boundary layer all the Reynolds stresses are much smaller than those expected in a corresponding boundary layer (having the same mean-velocity profile, say) developing on a plane surface. This implies that retention of only the major production term will not involve any appreciable error. The extra diffusion term in equation (21), which is expected to be generally smaller than the usual one, can be handled satisfactorily if it is assumed that $(\overline{\frac{uq^2}{2} + \frac{up'}{\rho}})$ is equal to $(\overline{\frac{vq^2}{2} + \frac{vp'}{\rho}})$, so that the same diffusion function can be used to model both terms. Experiments of Patel *et al.* further indicate that the ratio of $-\overline{uv}$ to $\overline{q^2}$ is nearly constant and equal to 0.15, the value found for thin boundary layers. If the above observations are introduced in equation (21) we obtain

$$\frac{1}{2a_1}\{U\frac{\partial\tau}{\partial x} + V\frac{\partial\tau}{\partial y}\} - \tau\frac{\partial U}{\partial y} + \frac{1}{r}\frac{\partial}{\partial x}(rG^*\frac{\tau}{a_1^{3/2}}\sqrt{\frac{\tau_{max}}{\rho}})$$

$$+ \frac{1}{r}\frac{\partial}{\partial y}(rG^*\frac{\tau}{a_1^{3/2}}\sqrt{\frac{\tau_{max}}{\rho}}) + \frac{1}{L^*}\frac{\tau^{3/2}}{\rho^{1/2}} = 0. \qquad (22)$$

Here it has been assumed that the diffusion and dissipation-length functions, $G^*$ and $L_D^* = L^*/a_1^{3/2}$, respectively, may be different from their thin boundary layer counterparts owing to the direct influence of transverse curvature on the turbulence. As suggested earlier, $L^*$ can be identified with the conventional mixing length if convection and diffusion are neglected. The variation of mixing length measured by Patel *et al.* in their thick boundary layer experiments is compared with the L function proposed by Bradshaw, Ferriss and Atwell (1967) in Figure 3. From this it is clear that the mixing length decreases markedly as the boundary layer thickness increases in relation to the local radius of curvature. The increase in dissipation implied by this, and the observed decrease in production mentioned earlier, suggest that the convection and diffusion of turbulent kinetic-energy, which are relatively unimportant in a thin boundary layer, become appreciable in a thick boundary layer. This, in turn, implies that it is no longer possible to associate the dissipation length with the conventional mixing length. Previous experience with the use of equation (20) in the calculation of thin boundary layers has shown that the dissipation length $L_D$ is the most important one of the three empirical functions (as might be expected from its association with mixing length). The observations made above, however, appear to suggest that in the treatment of thick boundary layers the diffusion function  and, to a lesser extent, the convective constant will play a greater role in the performance of equation (22) as a closure relation. Further work is obviously required to find the quantitative behavior of $G^*$ and $L^*$ across the boundary layer and their dependence on transverse curvature.

## IV. ON THE SOLUTION OF THE DIFFERENTIAL EQUATIONS

Regardless of the method used to close the differential equations of the mean flow, equations (15) through (17) remain elliptic. By contrast, the usual thin boundary-layer equations are either parabolic or hyperbolic depending on how the Reynolds stress is related to the mean flow. The ellipticity of the equations of the thick boundary layer means that it is not strictly possible to use conventional forward-marching numerical techniques. These equations have to be solved as a boundary value problem and herein lies a major difficulty since not all the boundary conditions are well defined.

The primary difficulty concerns the specification of the potential flow outside the boundary layer and the pressure distribution on the surface. For a thin boundary layer the constancy of static pressure across it simplifies the problem considerably since then the wall pressure distribution is simply related to the velocity in the freestream. For a thick boundary layer, however, the pressure remains an unknown quantity which we seek to determine. The y-momentum equation, which is usually ignored, now serves to relate the pressure field to the velocity field, but in order to solve it we need to prescribe either the pressure at the wall or that at the edge of the boundary layer. It is clear that neither is known a priori owing to the strong interaction between the boundary layer and the external potential flow. In order to proceed at all we therefore need some sort of an iterative scheme in which successive approximations are made for the external and boundary layer flows. One such procedure may involve the following steps: A potential flow solution may be obtained for the given axisymmetric body ignoring the boundary layer altogether. The pressure distribution on the wall so obtained can then be used to solve the x-momentum equation, the equation of continuity and the closure equation, ignoring the variation of static pressure, to obtain a first approximation for the boundary layer behavior. Notice that this destroys, artificially, the ellipticity of the mean flow equations. Since the first boundary layer calculation leads to the velocity distributions through the boundary layer, we can find the value of the stream function at the edge of the boundary layer. A second potential flow calculation can then be performed in which the condition of tangency of the flow on the body surface is replaced by the stream function at $y = \delta$ and some suitable extension of this stream function to infinity to account for the wake of the body. Thus, the second potential flow solution involves the stream function boundary condition extending to infinity along the edge of the boundary layer and the wake, and not the shape of the body as such. The pressure distribution along the edge of the boundary layer determined in this manner can then be used, together with the velocity field within the boundary layer obtained earlier, to find the static pressure variation across the boundary layer implied by the y-momentum equation. Note that this leads to a first approximation for the hitherto unknown pressure distribution on the wall. A second boundary layer calculation can then be performed using the pressure variation across the boundary layer thus obtained. Again, this involves the solution of the x-momentum equation, the equation of continutiy and the closure relation, so that the ellipticity of the complete set of equations is avoided. A number of iterations of this type will eventually lead not only to the prediction of the boundary layer development but also the pressure field associated with it. Perhaps the weakest link in this approach is the necessity to make some assumptions concerning the behavior of the wake, but it is expected that the potential flow at the edge of the boundary layer on the body will not be unduly sensitive to the precise assumptions that are made.

The amount of numerical computation involved in a procedure of the type described above is not too large when one considers the fact that methods for the solution of potential flow equations as well as the differential equations of the boundary layer are already in existence. The author and his colleagues have recently extended the method of Bradshaw, Ferriss and Atwell along the lines suggested in the previous section in order to calculate the development of a thick boundary layer when the static pressure variation across the boundary layer is prescribed. It is hoped to combine this method with a suitable potential flow calculation procedure to test the iteration scheme described above.

## V. MOMENTUM INTEGRAL EQUATION

In view of the difficulties associated with the solution of the differential equations of a thick axisymmetric boundary layer it may be more profitable to examine the possibilities of extending one or more of the well known approximate methods of calculation which are based on the integrated forms of the differential equations. If integral methods are to be considered it is necessary to obtain the momentum integral equation which includes the variation of static pressure across the boundary layer. This equation is readily obtained using standard procedures.

We write equation (15) in the form

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{dp_e}{dx} + \frac{1}{\rho}\frac{\partial}{\partial x}(p-p_e) + \frac{1}{r}\frac{\partial}{\partial y}\{r(\overline{uv}-\nu \frac{\partial U}{\partial y})\} = 0, \tag{23}$$

where the subscript e denotes the value at the edge of the boundary layer, i.e. at $y = \delta$. Then, using the equation of continuity, this can be rearranged to obtain

$$\frac{\partial}{\partial x}(U^2 r) - \frac{\partial}{\partial y}(U \int_o \frac{\partial}{\partial x}[Ur]dy) + \frac{r}{\rho}\frac{dp_e}{dx} + \frac{r}{\rho}\frac{\partial}{\partial x}(p-p_e)$$

$$+ \frac{\partial}{\partial y}\{r(\overline{uv}-\nu \frac{\partial U}{\partial y})\} = 0.$$

Integration of this with respect to y from the wall to the edge of the boundary layer gives

$$\frac{d}{dx}\int_o^\delta U^2 r dy - U_e \frac{d}{dx}\int_o^\delta U r dy + \frac{1}{\rho}\frac{dp_e}{dx}\int_o^\delta r dy + \int_o^\delta \frac{r}{\rho}\frac{\partial}{\partial x}(p-p_e)dy + \frac{\tau_w r_o}{\rho} = 0, \tag{24}$$

where $\tau_w = \mu(\frac{\partial U}{\partial y})_{y=o}$ is the wall shear stress.

If we now define the displacement thickness $\delta_1$, and the momentum thickness $\delta_2$, in forms appropriate to axisymmetric flow, viz

$$\delta_1 = \int_o^\delta (1 - \frac{U}{U_e}) \frac{r}{r_o} dy \ , \quad \delta_2 = \int_o^\delta \frac{U}{U_e}(1 - \frac{U}{U_e}) \frac{r}{r_o} dy, \tag{25}$$

and introduce the skin-friction coefficient

$$C_f = \frac{\tau_w}{\frac{1}{2} \rho U_e^2} , \tag{26}$$

equation (24) can be written

$$\frac{d\delta_2}{dx} + (2\delta_2+\delta_1) \frac{1}{U_e} \frac{dU_e}{dx} + \frac{\delta_2}{r_o} \frac{dr_o}{dx} - \frac{1}{2} C_f$$

$$= \frac{1}{U_e^2} \frac{d}{dx} (\frac{p_e}{\rho} + \frac{1}{2} U_e^2) \int_o^\delta \frac{r}{r_o} dy + \frac{1}{U_e^2} \int_o^\delta \frac{r}{r_o} \frac{\partial}{\partial x} (\frac{p-p_e}{\rho}) dy. \tag{27}$$

This is the basic form of the momentum integral equation for a thick axisymmetric boundary layer across which there is appreciable static pressure variation. The first term on the right hand side of this equation can be expressed in terms of the normal component of velocity at the edge of the boundary layer and the layer thickness by making use of the Bernoulli equation,

$$p_e + \frac{1}{2} \rho (U_e^2 + V_e^2) = \text{constant}, \tag{28}$$

which applies with sufficient accuracy at $y = \delta$, and the fact that

$$\int_o^\delta \frac{r}{r_o} dy = \delta \ \{1 + \frac{1}{2} \frac{\delta}{r_o} \cos\phi\}. \tag{29}$$

The second term on the right hand side of equation (27) represents simply the rate of change of the integrated pressure force across the boundary layer, and can be written in a number of different ways using the integrated y-momentum equation

$$p = p_e + \int_y^\delta (U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y}) dy = p_w - \int_o^y (U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y}) \ dy. \tag{30}$$

It will be noticed that, when the boundary layer is thin (i.e. when $p \approx p_e$ and $V_e << U_e$), both terms on the right hand side of equation (27) can be neglected, and the equation reduces to the well known form obtainable directly from the equations of thin axisymmetric boundary layers, namely equations (6) and (8).

## VI. CLOSURE OF THE INTEGRAL EQUATIONS

For a thin boundary layer the momentum integral equation contains three unknown, dimensionless, integral quantities: $R_\theta$, H and $C_f$, where $R_\theta \equiv U_e\delta_2/\nu$ is the momentum-thickness Reynolds number and $H \equiv \delta_1/\delta_2$ is the shape parameter of the velocity profile. When the velocity distribution in the external flow, or the pressure distribution on the wall, is specified this equation can be solved for $R_\theta$ only by providing two additional relationships. A large number of suggestions have been made for this purpose and most of these were examined at the Stanford Conference (Kline *et al.* 1968). Here, it suffices to note that closure of the momentum integral equation is usually effected by the introduction of a skin-friction formula of the form $C_f = C_f(H, R_\theta)$ and an auxiliary, or shape-parameter, equation which relates either directly or indirectly, the rate of change of H with x to the other variables in the momentum equation. The skin-friction formula and the shape-parameter equation often involve the explicit use of a velocity profile family. In what follows we shall explore the possibilities of extending some of these ideas to the treatment of thick axisymmetric boundary layers.

Examination of the momentum integral equation for the thick boundary layer obtained in the previous section shows that, even when the velocities and the pressure in the external flow are known, we have two additional unknowns, namely the boundary layer thickness and the integral involving the static pressure variation across the boundary layer. These unknowns are contained in the two terms on the right hand side of equation (27). The relative importance of these terms can best be judged by referring to experimental data. The recent measurements of Patel, Nakayama and Damian suggest that both terms are much larger than the other terms in the equation but the first is negative while the second is positive. The sum of the two, which is positive, however, is of the same order of magnitude as the other terms in the equation. In estimating the magnitude of the right hand side of equation (27) it is therefore advisable to regard it as a single term. In order to obtain some indication of the importance of these terms we have used the data of Patel *et al.* mentioned above. In Figure 4 the experimental values of $\delta_2$ are compared with those calculated from equation (27), with the right hand side omitted, using the measured values of $U_e$, $\delta_1$, and $C_f$. The disagreement between experiment and calculation indicates the importance of the terms which were omitted. Notice that these terms become appreciable only over the last ten percent of the body length where the boundary layer is thick and there is significant variation of static pressure across it. Various attempts were made to relate these terms with the other quantities in equation (27) but none of these proved very successful. An attempt was also made to evaluate these terms from the measured data but it was abandoned owing to the uncertainties involved in taking small differences between two large terms which themselves involved differentiation of ill-defined quantities such as the boundary layer thickness. During the course of these calculations, however, it was observed that the momentum thickness could be predicted accurately without including the right hand side of equation (27) if the second term on the left hand side of this equation was increased artificially. This could be accomplished quite simply by using the hypothetical freestream velocity $\bar{U}_e$ implied by the measured wall pressure distribution and Bernoulli's equation in place of the real measured variation of $U_e$. Thus, the momentum integral equation was approximated by

$$\left.\begin{aligned} &\frac{d\delta_2}{dx} + (2\delta_2 + \delta_1)\frac{1}{\bar{U}_e}\frac{d\bar{U}_e}{dx} + \frac{\delta_2}{r_o}\frac{dr_o}{dx} - \frac{1}{2}C_f = 0, \\ &\text{where} \qquad \rho\bar{U}_e\frac{d\bar{U}_e}{dx} = -\frac{dp_w}{dx}\,. \end{aligned}\right\} \qquad (31)$$

The results of integrating this equation using the measured values of $p_w$, $\delta_1$ and $C_f$ are also shown in Figure 4. From this we see that the usual thin boundary-layer momentum integral equation can be used to calculate the momentum thickness development with acceptable accuracy provided the usual pressure gradient term $\left(\frac{1}{U_e}\frac{dU_e}{dx}\right)$ is identified with the largest pressure gradient experienced by the boundary layer, namely the pressure gradient at the wall. This then represents an approximate but simple way in which the terms on the right hand side of equation (27) can be taken into account.

To use equation (31) in a calculation procedure we still need two additional relations. As remarked earlier, the skin-friction coefficient in a thin boundary layer is usually taken to be a function of H and $R_\theta$. From the experiments of Patel *et al.* it appears that even in the thick boundary layer the longitudinal velocity profiles conform well with the two-parameter families, such as those of Coles (1956), Thompson (1965), and others, from which the skin-friction formulae are deduced, provided the integral parameters are evaluated according to the thin plane-surface boundary layer definitions. Thus, the skin-friction law may be represented by

$$C_f = C_f(\overline{H}, \overline{R}_\theta), \tag{32}$$

where the bars denote "planar" definitions, i.e.

$$\overline{\delta_1} = \int_o^\delta (1 - \frac{U}{U_e})dy, \; \overline{\delta_2} = \int_o^\delta \frac{U}{U_e}(1 - \frac{U}{U_e})dy, \; \overline{R}_\theta = \frac{U_e \overline{\delta}_2}{\nu}, \; H = \frac{\overline{\delta}_1}{\overline{\delta}_2} \; . \tag{33}$$

It can be shown that the axisymmetric thicknesses defined by equation (25) can readily be related to the planar thicknesses above and the ratio $\delta/r_o$ when explicit use is made of a particular profile family, so that the skin-friction coefficient for the thick boundary layer can be expressed as a function of H, $R_\theta$ and $\delta/r_o$.

Considering the shape parameter equation next, it is clear that we can not possibly discuss the use of all the different equations proposed to date. Here we shall consider the well known entrainment method of Head (1958) since its extension to treat thick axisymmetric boundary layers can be made rather simply. For an axisymmetric boundary layer the volume flux, Q, at any streamwise location is

$$Q = \int_o^\delta 2\pi r \; U \; dy = 2\pi U_e\{r_o(\delta-\delta_1) + \frac{1}{2}\cos\phi \; \delta^2\}. \tag{34}$$

The rate of change of this with x is the rate of entrainment of freestream fluid into the boundary layer. Introducing a non-dimensional coefficient of entrainment, $C_E$, we have

$$\frac{1}{U_e r_o}\frac{d}{dx}\{U_e[r_o(\delta-\delta_1) + \frac{1}{2}\cos\phi \; \delta^2]\} = C_E. \tag{35}$$

When the boundary layer is thin (i.e. when $\delta \ll r_o$) this expression reduces to that given by Head. In order to make use of this equation in a calculation procedure we now need to make some assumption concerning $C_E$. For thin boundary layers Head postulated that $C_E$ depends upon the freestream velocity, a length scale of the flow in the outer region of the boundary layer and the shape of the velocity profile in this region, and from dimensional considerations deduced that $C_E$ is a function only of the shape parameter $H^* \equiv \frac{\delta-\delta_1}{\delta_2}$. Following similar logic we may generalize Head's result to consider thick axisymmetric boundary layers by assuming that $C_E$ is the same function of the shape parameter $\overline{H^*} \equiv \frac{\delta-\overline{\delta}_1}{\overline{\delta}_2}$ which reflects only the shape of the velocity profile without regard to the local radius of the surface. Whether this assumption is adequate, or we need to introduce an explicit dependence of $C_E$ on a curvature parameter such as $\delta/r_o$, can best be judged by detailed comparisons with experiment.

Equations (31), (32) and (35), together with the velocity profile family chosen to relate the planar definitions of the integral parameters to the conventional axisymmetric definitions, form a closed set in the three unknowns $C_f$, H and $R_\theta$. Although the boundary layer thickness $\delta$ appears explicitly in equation (35) and implicitly in the friction formula it is clear that it can be related to $\delta_2$ and the other integral parameters via the velocity profile family.

The method outlined above has been employed by the author to predict the development of thick axisymmetric turbulent boundary layers. The details of the method and its experimental verification are given in a separate paper (Patel 1973).

## VII. ON THE SOLUTION OF THE INTEGRAL EQUATIONS

To calculate the development of the boundary layer using the integral method of the previous section it is of course necessary to prescribe the pressure distribution on the wall. This, however, is not known a priori owing to the strong interaction between the thick boundary layer and the external potential flow. It is necessary therefore to resort once again to an iterative procedure in which successive approximations are made for the boundary layer and the potential flow. This aspect of the problem has already been discussed in detail in section V in conjunction with the solution of the differential equations. When integral equations are used for the boundary layer calculation the iterative procedure is identical but the solution of the y-momentum equation required to find the static pressure variation across the boundary layer will now involve the use of the velocity profile family.

## VIII. CONCLUDING REMARKS

Here we have considered the problem of the thick axisymmetric boundary layer near the tail of a body of revolution in some detail not only because of its practical importance in determining the form drag of such bodies but also because this is one of the few situations for which experimental data is available to guide our discussion. The case of a thick boundary layer developing on a cylinder of constant radius mentioned briefly in Section II has received rather more attention in the past. This problem is somewhat simpler insofar as the interaction between the boundary layer and the external potential flow may be assumed to be negligible on account of the simple geometry. Even in this case, however, it is likely that there is some direct influence of transverse curvature on the turbulence so that it may not be realistic to adopt closure relations established for thin boundary layers as has been done in the literature.

The problem of the interaction between the boundary layer and the external potential flow is of course not new. Such an interaction occurs in many situations and, for low-speed flows, has been studied in connection with the determination of the influence of thin boundary layers on presssure distributions on airfoils and other shapes. The calculation of the pressure distribution by potential flow theory after adding the displacement thickness of the boundary layer to the body surface is a well known example of the procedures used. The work described here differs from these conventional procedures in two respects. First, an attempt is made to show how the calculation methods used for thin boundary layers may be extended to consider thick boundary layers across which there is an appreciable variation of static pressure. Secondly, a procedure has been suggested for the calculation of the interaction between such a thick boundary layer and the external flow using the more realistic matching condition at the edge of the boundary layer.

In addition to the problem of obtaining better estimates for the drag of bodies of revolution the proposed method may also find application in the calculation of the flow in long conical and annular diffusers.

REFERENCES:

Bradshaw, P. 1969 "The Analogy Between Streamline Curvature and Buoyancy in Turbulent Shear Flow", *J. Fluid Mech., 36,* 177.

Bradshaw, P., Ferriss, D.H. and Atwell, N.P. 1967 "Calculation of Boundary-layer Development Using the Turbulent Energy Equation", *J. Fluid Mech., 28,* 593.

Cebeci, T. 1970 "Laminar and Turbulent Incompressible Boundary Layers on Slender Bodies of Revolution in Axial Flow", *J. Basic Eng., Trans. ASME, Ser. D, 92,* 545.

Coles, D. 1956 "The Law of the Wake in the Turbulent Boundary Layer", *J. Fluid Mech., 1,* 191.

Head, M.R. 1958 "Entrainment in the Turbulent Boundary Layer", British *Aeron. Res. Counc.,* R & M 3152.

Kelly, H.R. 1954 "A Note on the Laminar Boundary Layer on a Circular Cylinder in Axial Incompressible Flow", *J. Aeron. Sci., 21,* 634.

Kline, S.J., Morkovin, M.V., Sovran, G. and Cockrell, D.J. 1968 "Proceedings: Computation of Turbulent Boundary Layers- 1968 AFOSR-IFP-STANFORD CONFERENCE", Stanford University, California.

Nash, J.F. and Patel, V.C. 1972 "Three-Dimensional Turbulent Boundary Layers", *SBC Technical Book,* Atlanta.

Patel, V.C. 1973 "A Simple Integral Method for the Calculation of Thick Axisymmetric Turbulent Boundary Layers", To be published.

Patel, V.C., Nakayama, A. and Damian, R. 1973 "An Experimental Study of the Thick Turbulent Boundary Layer Near the Tail of a Body of Revolution", Iowa Institute of Hydraulic Research, Report No. 142.

Richmond, R.L. 1957 "Experimental Investigation of Thick Axially Symmetric Boundary Layers on Cylinders at Subsonic and Hypersonic Speeds", Ph.D. Thesis, California Inst. of Technology, Pasadena.

Seban, R.A. and Bond, R. 1951 "Skin Friction and Heat Transfer Characteristics of a Laminar Boundary Layer on a Cylinder in Axial Incompressible Flow", *J. Aeron. Sci., 18,* 671.

Stewartson, K. 1955 "The Asymptotic Boundary Layer on a Circular Cylinder", *Quart. Appl. Math., 13,* 113.

Thomspon, B.G.J. 1965 "A New Two-Parameter Family of Mean Velocity Profiles for Incompressible Turbulent Boundary Layers on Smooth Walls", British *Aeron. Res. Counc.,* R & M 3463.

Yasuhara, M. 1959 "Experiments of Axisymmetric Boundary Layers Along a Cylinder in Incompressible Flow", *Trans. Japan Soc. Aerospace Sci., 2,* 33.

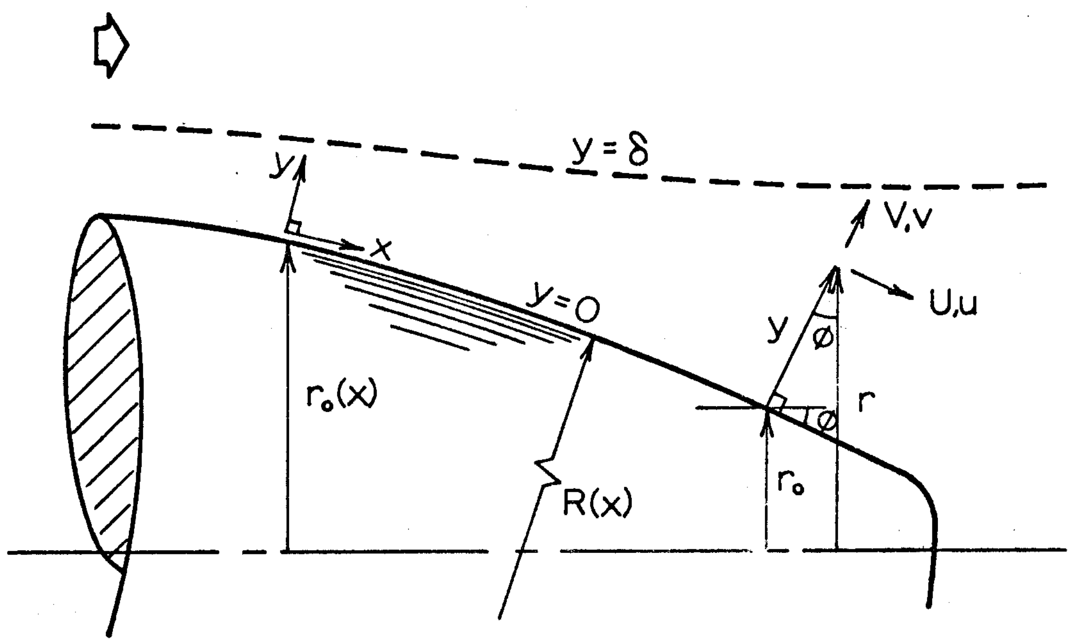
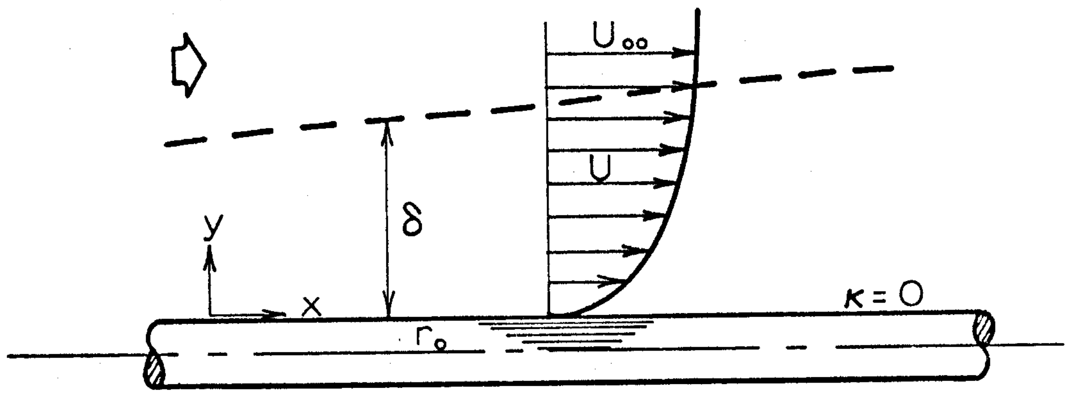
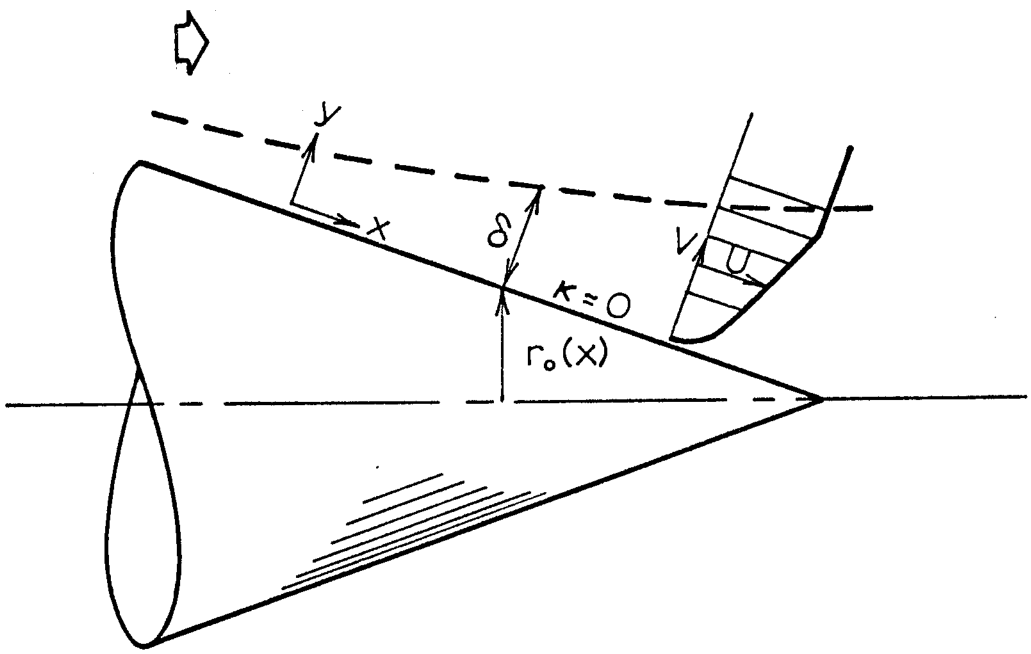



FIGURE 1. CO-ORDINATE SYSTEM AND NOTATION.

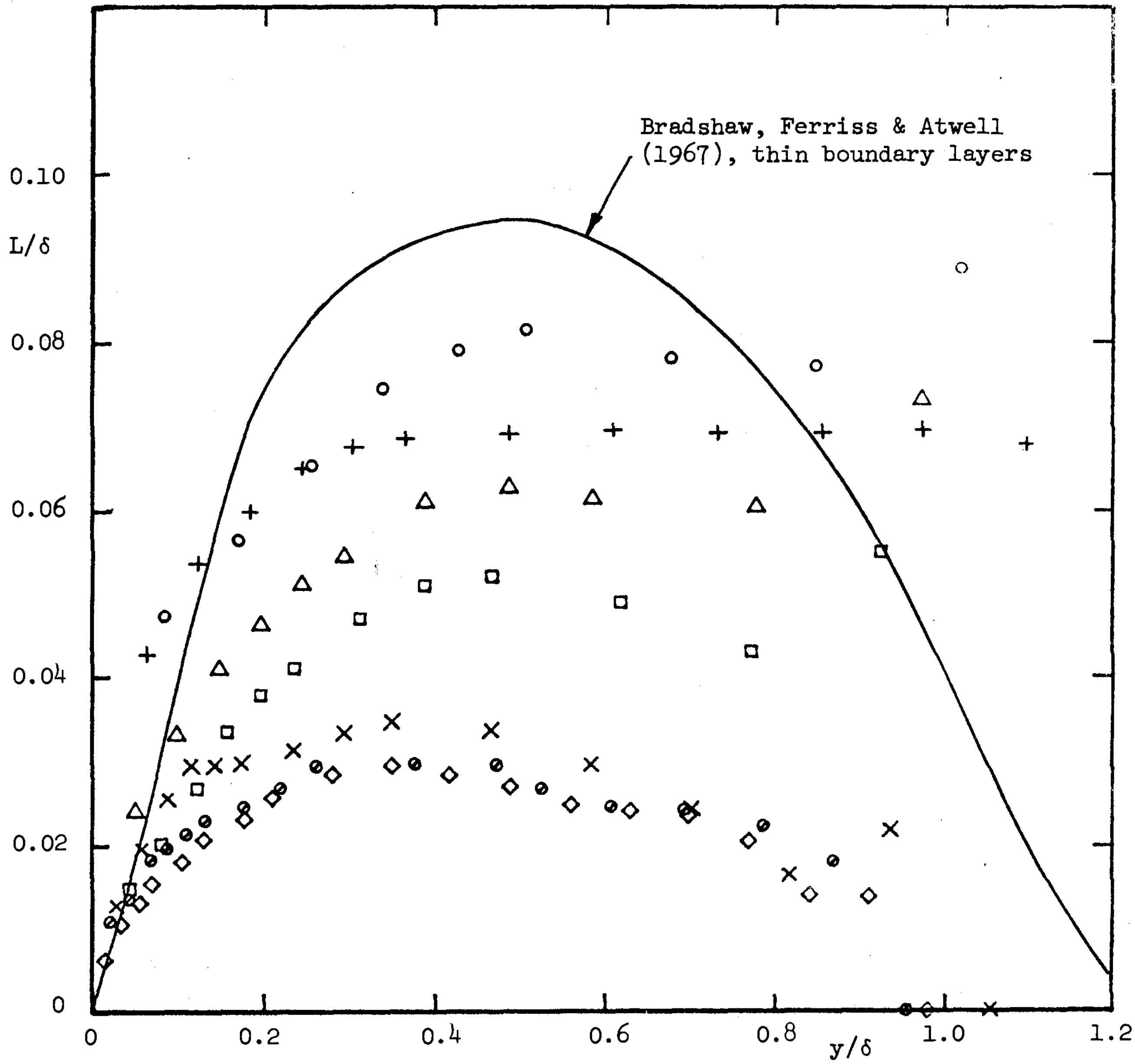



(a) Thick Axisymmetric Boundary Layer on a Long Slender Cylinder of Constant Radius.




(b) Thick Axisymmetric Boundary Layer Near the Conical Tail of a Body of Revolution.

FIGURE 2. TWO EXAMPLES OF FLOWS IN WHICH SIGNIFICANT TRANSVERSE CURVATURE EFFECTS ARE PRESENT.




| | X/L | $\delta/r_o$ | |
|---|---|---|---|
| ○ | 0.662 | 0.152 | Data of Patel, Nakayama and Damian (1973) |
| + | 0.80 | 0.261 | |
| △ | 0.85 | 0.381 | |
| □ | 0.90 | 0.619 | |
| × | 0.93 | 1.09 | |
| ⊘ | 0.96 | 2.56 | |
| ◇ | 0.99 | 13.00 | |

FIGURE 3. THE INFLUENCE OF TRANSVERSE CURVATURE ON MIXING LENGTH

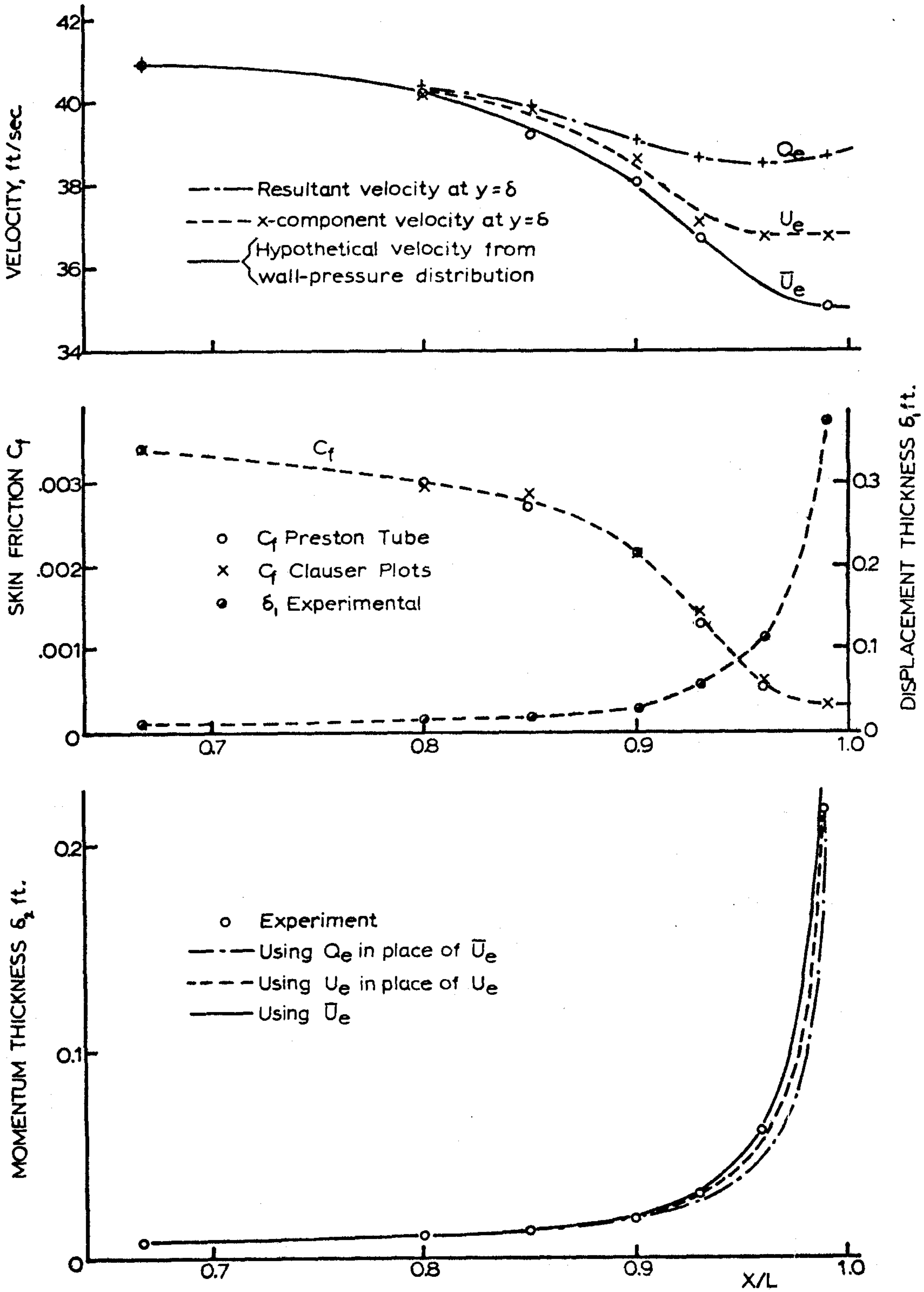



FIGURE 4. CALCULATION OF MOMENTUM THICKNESS FROM EQUATION (31) USING MEASURED VALUES OF $C_f$ AND $\delta_1$. EXPERIMENTS OF Patel, Nakayama and Damian (1973).

NSRDC LIST

March 19, 1973

40 Commander
Naval Ship Research and
Development Center
Bethesday, Maryland 20034
Attn: Code 1505
Code 5614 (39 cys)

1 Officer-in-Charge
Annapolis Laboratory
Naval Ship Research and
Development Center
Annapolis, Maryland 21402
Attn: Code 5642(Library)

6 Commander
Naval Ship Systems Command
Washington, D.C. 20360
Attn: SHIPS 2052 (3 cys)
SHIPS 03412B
SHIPS 0372
SHIPS 0342

12 Director
Defense Documentation Center
5010 Duke Street
Alexandria, Virginia 22314

1 Office of Naval Research
800 N. Quincy Street
Arlington, Virginia 22217
Attn: Mr. R.D. Cooper (Code 438)

1 Office of Naval Research
Branch Office
492 Summer Street
Boston, Mass. 02210

1 Office of Naval Research
Branch Office (493)
536 S. Clark Street
Chicago, Illinois 60605

1 Chief Scientist
Office of Naval Research
Branch Office
1030 E. Green Street
Pasadena, CA 91106

1 Office of Naval Research
Resident Representative
207 West 24th St.
New York, New York 10011

1 Office of Naval Research
Resident Representative
50 Fell Street
San Francisco, Ca 94102

2 Director
Naval Research Laboratory
Washington, D.C. 20390
Attn: Code 2027
Code 2629 (ONRL)

1 Commander
Naval Facilities Engineering
Command (Code 032C)
Washington, D.C. 20390

1 Library of Congress
Science & Technology Division
Washington, D.C. 20540

Commander
Naval Ordnance Systems
Command (ORD 035)
Washington, D.C. 20360

1 Commander
Naval Electronics Laboratory
Center (Library)
San Diego, Ca 92152

8 Commander
Naval Ship Engineering Center
Center Building
Prince Georges Center
Hyattsville, Maryland 20782
Attn: SEC 6034B
SEC 6110
SEC 6114H
SEC 6120
SEC 6136
SEC 6144G
SEC 6140B
SEC 6148

1 Naval Ship Engineering Center
Norfolk Division
Small Craft Engr. Dept.
Norfolk, Virginia 23511
Attn: D. Blount (6660.03)

1 Library (Code 1640)
Naval Oceanographic Office
Washington, D.C. 20390

1 Technical Library
Naval Proving Ground
Dehlgren, Virginia 22448

1 Commander (ADL)
Naval Air Development Center
Warminster, Penna 18974

1 Commanding Officer (L31)
Naval Civil Engineering Laboratory
Port Hueneme, CA 93043

1 Commander
Naval Undersea Center
San Diego, CA 92132
Attn: Dr. A. Fabula (6005)

2 Officer-in-Charge
Naval Undersea Center
Pasadena, CA 91107
Attn: Dr. J. Hoyt (2501)
Library (13111)

1 Director
Naval Research Laboratory
Underwater Sound Reference Division
P.O. Box 8337
Orlando, Flordia 32806

1 Library
Naval Underwater Systems Center
Newport, R.I. 02840

1 Research Center Library
Waterways Experiment Station
Corps of Engineers
P.O. Box 631
Vicksburg, Mississippi 39180

2 National Bureau of Standards
Washington, D.C. 20234
Attn: P. Klebanoff (FM 105)
Fluid Mechanics
Hydraulic Section

1 AFOSR/NAM
1400 Wilson Blvd.
Arlington, Virginia 22209

1 AFFOL/FYS (J. Olsen)
Wright Patterson AFB
Dayton, Ohio 45433

1 Dept. of Transportation
Library TAD-491.1
400 - 7th Street, S.W.
Washington, D.C. 20590

1 Boston Naval Shipyard
Planning Dept. Bldg 39
Technical Library, Code 202.2
Boston, Mass. 02129

1 Charleston Naval Shipyard
Technical Library
Naval Base
Charleston, S.C. 29408

1 Norfolk Naval Shipyard
Technical Library
Portsmouth, Virginia 23709

1 Philadelphia Naval Shipyard
Philadelphia, Penna 19112
Attn: Code 240

1 Portsmouth Naval Shipyard
Technical Library
Portsmouth, N.H. 03801

1 Puget Sound Naval Shipyard
Engineering Library
Bremerton, Wash. 98314

1 Long Beach Naval Shipyard
Technical Library (246L)
Long Beach, CA 90801

1 Hunters Point Naval Shipyard
Technical Library (Code 202.3)
San Francisco, CA 94135

1 Pearl Harbor Naval Shipyard
Code 202.32
Box 400, FPO
San Francisco, CA 96610

1 Mare Island Naval Shipyard
Shipyard Technical Library
Code 202.3
Vallejo, CA 94592

1 Assistant Chief Design Engineer
for Naval Architecture (Code 250)
Mare Island Naval Shipyard
Vallejo, CA 94592

3 U.S. Naval Academy
Annapolis, Maryland 21402
Attn: Technical Library
Dr. Bruce Johnson
Prof. P. Van Mater, Jr.

3 Naval Postgraduate School
Monterey, CA 93940
Attn: Library, Code 2124
Dr. T. Sarpkaya
Prof. J. Miller

1 Capt. L.S. McCready, USMS
Director, National Maritime
Research Center
U.S. Merchant Marine Academyy
Kings Point, L.I., New York 11204
Attn: Academy Library

1 U.S. Merchant Marine Academy
Kings Point, L.I., New York 11204
Attn: Academy Library

1 Library
The Pennsylvania State University
Ordnance Research Laboratory
P.O. Box 30
State College, Penna 16801

1 Bolt, Beranek & Newman
1501 Wilson Blvd.
Arlington, Virginia 22209
Attn: Dr. F. Jackson

1 Bolt, Beranek & Newman
50 Moulton Street
Cambridge, Mass. 02138
Attn: Library

1 Bethlehem Steel Corporation
Center Technical Division
Sparrows Point Yard
Sparrows Point, Maryland 21219

1 Bethlehem Steel Corporation
25 Broadway
New York, New York 10004
Attn: Library (Shipbuilding)

Cambridge Acoustical Associates, Inc.
1033 Mass Avenue
Cambridge, Mass. 02138
Attn: Dr. M. Junger

1 Cornell Aeronautical Laboratory
Aerodynamics Research Dept.
P.O. Box 235
Buffalo, New York 14221
Attn: Dr. A. Ritter

1 Eastern Research Group
P.O. Box 222
Church Street Station
New York, New York 10008

1 Esso International
Design Division, Tanker Dept.
15 West 51st Street
New York, New York 10019

1 Mr. V. Boatwright, Jr.
R & D Manager
Electric Boat Division
General Dynamics Corporation
Groton, Conn. 06340

1 Gibbs & Cox, Inc.
21 West Street
New York, New York 10006
Attn: Technical Info. Control

1 Hydronautics, Inc.
Pindell School Road
Howard County
Laurel, Maryland 20810
Attn: Library

2 McDonnell Douglas Aircraft Co.
3855 Lakewood Blvd.
Long Beach, CA 90801
Attn: J. Hess
A.M.O. Smith

1 Lockheed Missiles & Space Co.
P.O. Box 504
Sunnyvale, CA 94088
Attn: Mr. R.L. Waid, Dept. 57-74
Bldg. 150, Facility 1

1 Newport News Shipbuilding & Dry
Dock Company
4101 Washington Avenue
Newport News, Virginia 23607
Attn: Technical Library Dept.

1 North American Aviation, Inc.
Space & Information Systems Div.
12214 Lakewood Blvd.
Downey, CA 90241
Attn: Mr. Ben Ujihara (SL-20)

1 Nielsen Engineering & Research, Inc.
850 Maude Avenue
Mountain View, CA 94040
Attn: Mr. S.B. Spangler

1 Oceanics, Inc.
Technical Industrial Park
Plainveiw, L.I., New York 11803

1 Society of Naval Architects
and Marine Engineers
74 Trinity Place
New York, New York 10006
Attn: Technical Library

1 Sun Shipbuilding & Dry Dock Co.
Chester, Penna 19000'
Attn: Chief Naval Architect

1 Sperry Systems Management Division
Sperry Rand Corporation
Great Neck, New York 11020
Attn: Technical Library

1 Stanford Research Institute
Menlo Park, CA 94025
Attn: Library G-021

2 Southwest Research Institute
P.O. Drawer 28510
San Antonio, Texas 78284
Attn: Applied Mechanics Review
Dr. H. Abramson

1 Tracor, Inc.
6500 Tracor Lane
Austin, Texas 78721

1 Mr. Robert Taggart
3930 Walnut Street
Fairfax, Virginia 22030

1 Ocean Engr. Department
Woods Hole Oceanographic Inst.
Woods Hole, Mass. 02543

1 Worcester Polytechnic Inst.
Alden Research Laboratories
Worcester, Mass. 01609
Attn: Technical Library

1 Applied Physics Laboratory
University of Washington
1013 N. E. 40th Street
Seattle, Washington 98105
Attn: Technical Library

1 University of Bridgeport
Bridgeport, Conn. 06602
Attn: Dr. E. Uram

1 Cornell University
Graduate School of Aerospace Engr.
Ithaca, New York 14850
Attn: Prof. W.R. Sears

4 University of California
Naval Architecture Department
College of Engineering
Berkeley, CA 94720
Attn: Library
Prof. W. Webster
Prof. J. Paulling
Prof. J. Wehausen

3 California Institute of Technology
Pasadena, CA 91109
Attn: Aeronautics Library
Dr. T.Y. Wu
Dr. A.J. Acosta

1 Docs/Repts/Trans Section
Scripps Institution of
Oceanography Library
University of California, San Diego
P.O. Box 2367
La Jolla, CA 92037

1 Catholic University of America
Washington, D.C. 20017
Attn: Dr. S. Heller, Dept. of
of Civil & Mech. Engr.

1 Colorado State University
Foothills Campus
Fort Collins, Colorado 80521
Attn: Reading Room, Engr. Res. Center

1 University of Calif. at San Diego
La Jolla, CA 92038
Attn: Dr. A.T. Ellis
Dept. of Applied Math

2 Flordia Atlantic University
Ocean Engineering Department
Boca Raton, Fla 33432
Attn: Technical Library
Dr. S. Dunne

2 Harvard University
Pierce Hall
Cambridge, Mass 02138
Attn: Prof. G. Carrier
Gordon McKay Library

1 University of Hawaii
Department of Ocean Engineering
2565 The Mall
Honolulu, Hawaii 96822
Attn: Dr. C. Bretschneider

1 University of Illinois
Urbana, Illinois 61801
Attn: Dr. J. Robertson

3 Institute of Hydraulic Research
The University of Iowa
Iowa City, Iowa 52241
Attn: Library
Dr. L. Landweber
Dr. J. Kennedy

1 The John Hopkins University
Baltimore, Md. 21218
Attn: Prof. O. Phillips
Mechanics Depts.

1 Kansas State University
Engineering Experiment Station
Seaton Hall
Manhattan, Kansas 66502
Attn: Prof. D. Nesmith

1 University of Kansas
Chm. Civil Engr. Dept. Library
Lawrence, Kansas 66044

1 Fritz Engr. Laboratory Library
Depart. of Civil Engr.
Lehigh University
Bethlehem, Penna 18015

5 Department of Ocean Engineering
Massachusetts Inst. of Technology
Cambridge, Mass. 02139
Attn: Department Library
Prof. P. Leehey,
Profs. M. Abkowitz , P. Mandel
Dr. J. Newman

1 Parsons Laboratory
Massachusetts Institute of Technology
Cambridge, Mass 02139
Attn: Prof. A. Ippen

5 St. Anthony Falls Hydr. Laboratory
University of Minnesota
Mississippi River at 3rd Ave., S.E.
Minneapolis, Minn. 55414
Attn: Director
Mr. J. Wetzel
Mr. F. Schiebe
Mr. J. Killen
Dr. C. Song

3 Department of Naval Architecture
and Marine Engineering
University of Michigan
Ann Arbor, Michigan 48104
Attn: Library
Dr. T.F. Ogilvie
Prof. F. Hammitt

2 College of Engineering
University of Notre Dame
Notre Dame, Indiana 46556
Attn: Engineering Library
Dr. A. Strandhagen

2 New York University
Courant Inst. of Math. Sciences
251 Mercier Street
New York, New York 10012
Attn: Prof. A. Peters
Prof. J. Stoker

New York University
University Heights
Bronx, New York 10453
Attn: Prof. W. Pierson, Jr.

Department of Aerospace and
Mechanical Sciences
Princeton University
Princeton, New Jersey 08540
Attn: Prof. G. Mellor

3 Davidson Lab
Stevens Institute of Technology
711 Hudson Street
Hoboken, New Jersey 07030
Attn: Library
Mr. J. Breslin
Mr. S. Tsakonas

1 Department of Mathematics
St. John's University
Jamaica, New YORK 11432
Attn: Prof. J. Lurye

1 Applied Research Lab Library
University of Texas
P.O. Box 8029
Austin, Texas 78712

College of Engineering
Utah state University
Logan, Utah 84321
Attn: Dr. R. Jeppson

2 Stanford University
Stanford, CA 94305
Attn: Engineering Library
Dr. R. Street

3 Webb Institute of Naval Architecture
Crescent Beach Road
Glen Cover, L.I., New York 11542
Attn: Library
Prof. E.V. Lewis
Prof. L.W. Ward

1 National Science Foundation
Engineering Division Library
1800 G Street N.W.
Washington, D.C. 20550

1 University of Connecticut
Box U-37
Storrs, Conn. 06268
Attn: Dr. V. Scottron
Hydraulic Res. Lab

Long Island University
Graduate Department of
Marine Science
40 Merrick Avenue
East Meadow, L.I., N.Y. 11554
Attn: Prof. David Price

1 Dr. Douglas E. Humphreys (Code 712)
Naval Coastal Systems Lab
Panama City, Florida 32401

Unclassified

Security Classification

**DOCUMENT CONTROL DATA - R & D**

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

| | | |
|---|---|---|
| 1. ORIGINATING ACTIVITY (Corporate author) | 2a. REPORT SECURITY CLASSIFICATION | |
| Institute of Hydraulic Research<br>The University of Iowa<br>Iowa City, Ia. | Unclassified<br>2b. GROUP | |
| 3. REPORT TITLE | | |
| "On the Equations of a Thick Axisymmetric Turbulent Boundary Layer" | | |
| 4. DESCRIPTIVE NOTES (Type of report and inclusive dates) | | |
| IIHR Report No. 143 | | |
| 5. AUTHOR(S) (First name, middle initial, last name) | | |
| V.C. Patel | | |
| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
| January 1973 | 28 | 15 |
| 8a. CONTRACT OR GRANT NO.<br>N00014-68-A-0196-0002<br>b. PROJECT NO.<br>c.<br>d. | 9a. ORIGINATOR'S REPORT NUMBER(S)<br>IIHR Report No. 143<br>9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) | |
| 10. DISTRIBUTION STATEMENT | | |
| Approved for public release; distribution unlimited | | |
| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY<br>Naval Ship Research & Development Center | |

13. ABSTRACT

From an examination of the Reynolds equations for axisymmetric turbulent flow in situations where the thickness of the boundary layer is of the same order as the transverse radius of curvature of the surface, it has been shown that, in general, neither the boundary layer nor the potential flow outide it can be calculated independently of the other, owing to significant interaction between the two flow regimes. Following a discussion of the various procedures for extending conventional thin boundary-layer calculation methods to treat thick axisymmetric turbulent boundary-layers, taking into account the influence of transverse curvature either at the differential or the integral level, a method is proposed for the simultaneous solution of the boundary layer and the potential flow equations, allowing the two flow regimes to interact.

Unclassified

Security Classification

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Turbulent Boundary Layers | | | | | | |
| Transverse Curvature | | | | | | |
| Interaction | | | | | | |
| Calculation methods | | | | | | |
| Differential equations | | | | | | |
| Integral equations | | | | | | |
| Reynolds stresses | | | | | | |