ORTHOGONAL COORDINATE SYSTEMS FOR THREE-DIMENSIONAL BOUNDARY LAYERS, WITH PARTICULAR REFERENCE TO SHIP FORMS

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T. Miloh and V. C. Patel

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Iowa Institute of Hydraulic Research
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ABSTRACT

The problem of choosing orthogonal, curvilinear, coordinate systems for use in boundary-layer calculations on arbitrary three-dimensional bodies is considered in some detail. A general method for the practical evaluation of the various geometrical properties of the coordinates occurring in the three-dimensional boundary-layer equations is described. A particular coordinate system which appears to be the most convenient one for ship hulls is then proposed and analyzed further.

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ORTHOGONAL COORDINATE SYSTEMS FOR THREE-DIMENSIONAL BOUNDARY LAYERS, WITH PARTICULAR REFERENCE TO SHIP FORMS

I. INTRODUCTION

In the calculation of laminar and turbulent boundary layers on arbitrary three-dimensional bodies, such as ship forms or shapes of aerodynamic interest, an important problem that has to be resolved first is the choice of a convenient coordinate system. For reasons of simplicity the Navier-Stokes equations are usually written in a curvilinear coordinate system that is orthogonal everywhere in space, the choice of a non-orthogonal system, although legitimate, being too awkward. In order to facilitate the application of the no-slip boundary condition at the wall it is convenient to choose, for the Navier-Stokes equations as well as for boundary layer calculations, a system of coordinates such that the surface of the body is one member of a family of coordinate surfaces. The theory of differential geometry then shows that there is one, and only one, coordinate system which satisfies the requirements of orthogonality. In this unique system the coordinate surfaces normal to the body must intersect the body surface in its lines of principal curvature. This is often called the theorem of Dupin (see, for example, Struik, 1950). To find this coordinate system for a body of given shape it is necessary therefore to obtain the lines of curvature of the body surface.

The equations of three-dimensional boundary layers are derived usually from the Navier-Stokes equations written in the triply-orthogonal curvilinear coordinate system by making the assumption that the thickness of the boundary layer is everywhere an order of magnitude smaller than the principal radii of the body surface. An important simplification that arises from the introduction of the boundary layer approximations, as pointed out first by Howarth (1951), is that it is no longer necessary to employ a coordinate system that is orthogonal everywhere in space.

Thus, for the treatment of thin boundary layers it suffices to choose a coordinate system that is orthogonal locally on the body surface but not necessarily so elsewhere. For a body of given geometry there are of course a large number of coordinate systems which will satisfy this requirement.

A coordinate system that has been found to be particularly useful in the calculation of three-dimensional boundary layers employing integral methods is the so-called streamline or intrinsic coordinate system. Here, the coordinates are chosen to coincide with the projections of the external potential flow streamlines on the body surface and their orthogonal trajectories. Thus, the coordinates are geared to the potential flow outside the boundary layer, and in order to proceed with the boundary layer calculation it is necessary first to determine the shape of the streamlines in the potential flow. Quite apart from the shortcomings of the integral methods which are wedded to the use of streamline coordinates, there are a number of reasons which make the use of such a coordinate system inconvenient from a practical standpoint. First of all, as pointed out by Landweber (1971), the solution of the potential flow about arbitrary three-dimensional shapes is a task of the same order of magnitude as the boundary-layer calculations themselves, and for many practical situations, such as the floating ship problem, adequate methods for the calculation of potential flow simply do not exist. Secondly, even if such methods were available, a great many potential flow calculations, and boundary layer calculations in corresponding streamline coordinates, will need to be performed in order to make a useful parametric study. Thus, for example, in the case of a floating ship, a large number of different coordinate systems will be needed to study the influence of such parameters as the Reynolds number, the Froude number, the trim angle and the draft-length ratio. In view of these difficulties it appears to be much more practical to select a coordinate system which depends only upon the geometry of the body rather than on the potential flow.

As stated earlier, there are many coordinate systems which satisfy the requirement of local orthogonality on the body surface. Given a body of particular shape there will invariably be a system which is more convenient than others. Leaving the choice of a particular coordinate system aside for a moment, and examining the boundary layer equations (see, for example, Nash and Patel, 1972), we find that in order to proceed with the solution of these equations it is first necessary to obtain certain geometrical properties of the coordinate system. These are usually the geodesic curvatures of the orthogonal lines chosen on the body surface, the linearizing factors or "metric coefficients" associated with these lines, and the radii of curvature of the body surface along these lines. It should be emphasized that such calculations have to be performed regardless of what coordinate system is finally chosen. The main purpose of this paper is to present a general method for the determination of the various geometrical properties with particular reference to arbitrary ship forms. A coordinate system which appears to be the most convenient one for calculating ship boundary layers is then suggested and discussed in greater detail.

Mathematically, the general problem may be formulated as follows: The equation of an arbitrary three-dimensional surface is given in the form

$$F(x,y,z) = 0 ,$$

where x,y,z are Cartesian coordinates referred to some origin. On this surface is drawn a curved line, say ξ = constant. This line does not necessarily coincide with a line of principal curvature of the surface. The problem then is to find the two curvatures of this line, namely the geodesic curvature K_g in the tangent plane to the surface, and the normal curvature K_n which is the curvature of the surface along the line ξ = constant. Once a procedure for calculating these curvatures is established we can use it to obtain all the necessary information for any orthogonal net of lines drawn on the surface. In particular, if the family of lines ξ = constant, and their orthogonal trajectories η = constant say, are chosen to coincide with the lines of principal curvature of the surface, then the normal curvatures of these lines are simply the principal curvatures of the surface. Furthermore, if an expression is obtained for the normal curvature of an arbitrary line drawn on the surface, it is possible to obtain from it the equations of the lines of principal curvature of the surface since, by definition, these are lines along which K_n is either a maximum or a minimum. Thus, the solution to the general problem outlined above gives, in addition to the information concerning any orthogonal net drawn on the surface required in thin boundary-layer calculations, a method for the determination of the unique triply-orthogonal coordinate system which has to be used when the usual thin boundary-layer approximations do not apply. Such a situation may arise, for example, in the neighborhood of the stern of a ship where the boundary layer usually grows to a thickness comparable with the local radii of curvature of the surface.

In order to make the results of this study directly applicable to ship boundary layers we shall develop the theory for a slightly restrictive class of surfaces. The more general case can be treated in a similar manner. Here, it is assumed that the surface is prescribed by an equation of the form

$$F(x,y,z) = y - f(x,z) = 0.$$

Expressions are then obtained for the geodesic and normal curvatures of an arbitrary line drawn on the surface. These are used to find the lines of curvature and the principal radii of curvature of the surface. General formulae are also obtained for the metric coefficients associated with any set of orthogonal lines drawn on the surface.

A particular coordinate system that appears to be the most convenient one for ship boundary-layer work is then described. It is shown that, for this system, the geodesic and normal curvatures, as well as the metric coefficients, of the coordinate lines on the surface can all be expressed in terms of the various first and second derivatives of f. A numerical procedure for evaluating these derivatives for arbitrary ship forms is considered next. This involves the use of spline-approximation techniques. The general usefulness of the method is demonstrated by taking some simple examples.

II. NORMAL AND GEODESIC CURVATURES

In this section we shall obtain expressions for the normal and geodesic curvatures of an arbitrary line drawn on the three-dimensional surface defined by the equation

$$y = f(x,z). \tag{1}$$

If equation (1) is used to prescribe a ship hull, then x is directed from bow to stern, z is positive upwards, and y = 0 is the vertical plane of symmetry, i.e. the centerplane.

A three-dimensional curve is represented in a parametric form by

$$x = x(t) \; ; \quad y = y(t) \; ; \quad z = z(t) \; ; \tag{2}$$

where t is an arbitrary parameter. A radius vector from the origin to a point on this curve is then

$$\vec{R} = \vec{i}x(t) + \vec{j}y(t) + \vec{k}z(t) \; , \tag{3}$$

where $\vec{i},\vec{j},\vec{k}$ are unit vectors in the directions x,y,z respectively. A unit tangent to the curve is given by

$$\vec{T} = \frac{\frac{d\vec{R}}{dt}}{\left|\frac{d\vec{R}}{dt}\right|} = \frac{\vec{i}\dot{x} + \vec{j}\dot{y} + \vec{k}\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \; , \tag{4}$$

where the dot symbol represents differentiation with respect to t. The curvature of the curve is defined by

$$\vec{K} = \frac{d\vec{T}}{ds} \; , \tag{5}$$

where s denotes the arc length along the curve. The curvature is a vector in the direction of the principal normal to the curve.

We now denote the unit normal to the surface on which the curve lies by $\vec{N}$, and a unit vector perpendicular to $\vec{T}$ and $\vec{N}$ by $\vec{Q}$, so that $\vec{Q} = \vec{T} \times \vec{N}$. Thus, $\vec{Q}$ is a unit vector in the direction of the tangent to the surface which intersects orthogonally with the original curve. The projection of the curvature vector $\vec{K}$ on the surface normal $\vec{N}$ is, according to Meusnier's theorem (Struik, 1950), the curvature of the normal section in the direction of $\vec{N}$; this curvature is denoted by K_n and is no more than the curvature of the surface along the original curve drawn on the surface. The projection of the curvature vector on $\vec{Q}$ is usually referred to as the geodesic curvature of the original curve and is denoted by K_g. Thus, we have

$$K_n = \vec{K} \cdot \vec{N} \tag{6}$$

and

$$K_g = \vec{K} \cdot (\vec{T} \times \vec{N}) \quad . \tag{7}$$

Equation (4) may also be written as

$$\vec{T} = t'(\vec{i}\dot{x} + \vec{j}\dot{y} + \vec{k}\dot{z}) \quad , \tag{8}$$

where the prime denotes differentiation with respect to the arc length s, so that

$$t' = \frac{dt}{ds} = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{-\frac{1}{2}} \quad . \tag{9}$$

Substitution of these relations in equation (5) results in the following expression for the curvature vector:

$$\vec{K} = \vec{i}[(t')^2\ddot{x} + t''\dot{x}] + \vec{j}[(t')^2\ddot{y} + t''\dot{y}] + \vec{k}[(t')^2\ddot{z} + t''\dot{z}] \quad . \tag{10}$$

Now, a unit vector normal to the surface defined by equation (1) is

$$\vec{N} = D(\vec{i}\, f_x - \vec{j} + \vec{k}\, f_z) \quad , \tag{11}$$

where

$$D = (1 + f_x^2 + f_z^2)^{-\frac{1}{2}} \tag{12}$$

and subscripts on f denote partial derivatives.

The normal and geodesic curvatures are obtained when equations (1), (10) and (11) are substituted into equations (6) and (7). The final expressions are

$$K_n = -\frac{D\,\mathrm{II}}{\mathrm{I}} \tag{13}$$

and

$$K_g = -\frac{(\dot{x}\ddot{z} - \dot{z}\ddot{x}) + \mathrm{II}\,D^2(\dot{x}f_z - \dot{z}f_x)}{D(\mathrm{I})^{3/2}} \ , \tag{14}$$

where

$$\mathrm{II} = \dot{x}^2 f_{xx} + 2\dot{x}\dot{z}\, f_{xz} + \dot{z}^2 f_{zz} \tag{15}$$

and

$$\mathrm{I} = \dot{x}^2(1 + f_x^2) + 2\dot{x}\dot{z}\, f_x f_z + \dot{z}^2(1 + f_z^2) \quad . \tag{16}$$

Here, I and D II represent the first and second fundamental forms of differential geometry.

It is of interest to note the sign convention for the two curvatures. The normal curvature is defined to be positive when it is directed along the outward normal to the surface. The geodesic curvature of a line is positive when it is directed along the positive tangent direction of the orthogonal line.

The results obtained in this section are of course applicable to any arbitrary curve drawn on the surface $y = f(x,z)$. As mentioned in the Introduction, we may work with the equations of a thin boundary layer written in any system of coordinates provided the coordinates form an orthogonal net on the body surface. If we choose a family of curves $\xi(x,z)$ = constant, and their orthogonal trajectories $\eta(x,z)$ = constant, to form such a net, then the curvature terms which occur in the boundary layer equations can all be found at every point on the surface by applying equations (13) and (14) to each member of the two families of curves. In order to complete the geometrical description of the coordinate system and proceed with the solution of the equations there remains only to find the linearizing factors, or metric coefficients, h_ξ and h_η, associated with the ξ and η lines. A general procedure for determining these is described in the following section.

III. METRIC COEFFICIENTS OF ORTHOGONAL LINES ON THE SURFACE

If we denote an element of length measured on the surface $y = f(x,z)$ along the line $\eta(x,z)$ = constant by ds_ξ, and that along the line $\xi(x,z)$ = constant by ds_η, we have, by the definition of the metric coefficients h_ξ and h_η,

$$(ds_\xi)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (h_\xi d\xi)^2 \tag{17}$$

and

$$(ds_\eta)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (h_\eta d\eta)^2 . \tag{18}$$

Now, along η = constant we have

$$\eta_x dx + \eta_z dz = 0 , \qquad \xi_x dx + \xi_z dz = d\xi ,$$

so that

$$dx = \frac{\eta_z}{\xi_x \eta_z - \xi_z \eta_x} d\xi , \qquad dz = \frac{\eta_x}{\xi_z \eta_x - \xi_x \eta_z} d\xi . \tag{19}$$

Also, from equation (1)

$$dy = f_x dx + f_z dz . \tag{20}$$

Substitution of equations (19) and (20) in equation (17) leads to

$$h_\xi^2 = \frac{\eta_x^2 + \eta_z^2 + (f_x \eta_z - f_z \eta_x)^2}{(\xi_x \eta_z - \xi_z \eta_x)^2} . \tag{21}$$

A similar analysis applied to the ξ = constant line gives

$$h_\eta^2 = \frac{\xi_x^2 + \xi_z^2 + (f_x \xi_z - f_z \xi_x)^2}{(\xi_x \eta_z - \xi_z \eta_x)^2} . \tag{22}$$

Thus, the metric coefficients can be found once the equations of the ξ and η lines are given.

We may note here that the geodesic curvatures of the orthogonal ξ,η lines on the surface may also be deduced from the metric coefficients since the theory of differential geometry shows that

$$(K_g)_\xi = \text{constant} = \frac{1}{h_\xi h_\eta} \frac{\partial h_\eta}{\partial \xi} \tag{23}$$

and

$$(K_g)_\eta = \text{constant} = \frac{1}{h_\xi h_\eta} \frac{\partial h_\xi}{\partial \eta} \ . \tag{24}$$

While this provides a useful check on the expressions for the geodesic curvatures it should be mentioned that the general method presented in Section II is to be preferred since it also gives the normal curvatures without any reference to the metric coefficient associated with the direction normal to the surface.

IV. LINES OF PRINCIPAL CURVATURE

The lines of principal curvature of an arbitrary surface are a set of orthogonal lines on the surface along which the normal curvature takes extreme values, maximum value along one set and minimum value along the other. For surfaces which are given in the form of equation (1), it is easier to find the lines of principal curvature by first finding their projections on the surface $y = 0$. This is easily accomplished by choosing $t = x$, so that $\dot{x} = 1$ and $\dot{z} = dz/dx$, and setting the derivative of K_n with respect to $\dot{z}$ equal to zero. When such an operation is performed on equation (13) one obtains the following quadratic in $\dot{z}$:

$$[f_x f_z f_{zz} - f_{xz}(1 + f_z^2)]\dot{z}^2 + [f_{zz}(1 + f_x^2) - f_{xx}(1 + f_z^2)]\dot{z} + [f_{xz}(1 + f_x^2) - f_x f_z f_{xx}] = 0 \ . \tag{25}$$

The two solutions of this quadratic define the projections of the lines of curvature on the $y = 0$ plane. Use of equation (1) then gives the lines of curvature on the surface $y = f(x,z)$. Equation (25) has also been obtained by Landweber (1971) by using a slightly different approach.

The normal and geodesic curvatures of the lines of principal curvature are given by

$$K_n = -\frac{(f_{xx} + 2\dot{z}f_{xz} + \dot{z}^2 f_{zz})}{(1 + f_x^2 + f_z^2)^{\frac{1}{2}}\,[1 + f_x^2 + 2\dot{z}f_x f_z + \dot{z}^2(1 + f_z^2)]} \qquad (26)$$

and

$$K_g = -\frac{\ddot{z} + (1 + f_x^2 + f_z^2)\,(f_z - \dot{z}f_x)\,(f_{xx} + 2\dot{z}f_{xz} + \dot{z}^2 f_{zz})}{(1 + f_x^2 + f_z^2)^{\frac{1}{2}}\,[1 + f_x^2 + 2\dot{z}f_x f_z + \dot{z}^2(1 + f_z^2)]^{3/2}}\,, \qquad (27)$$

where $\dot{z}$ represents the two solutions of equation (25) and $\ddot{z} = \frac{d\dot{z}}{dx}$. Note that equation (26) gives the principal curvatures of the surface $y = f(x,z)$.

If the lines of principal curvature are used as the coordinate system, the metric coefficients associated with them can be obtained from equations (21) and (22) by substituting in them $\frac{\xi_x}{\xi_z} = -\dot{z}_1$ and $\frac{\eta_x}{\eta_z} = -\dot{z}_2$, where $\dot{z}_1$ and $\dot{z}_2$ are the two solutions of equation (25).

V. A COORDINATE SYSTEM FOR SHIP BOUNDARY LAYERS

In this section we shall consider in somewhat greater detail a coordinate system which promises to be the most convenient one for the calculation of three-dimensional boundary layers on ship forms. The shape of a ship hull is usually specified in the form of equation (1), viz

$$y = f(x,z),$$

where, as mentioned before, y is measured from the centerplane, x is directed from bow to stern, and z is measured vertically upwards. In the theoretical analysis of potential flow about ship forms it is customary to assume some algebraic relation for f but in practice, for prototype ships and models, f is usually prescribed numerically in the form of tables of y versus z at each streamwise position x. Furthermore, when experimental data on pressure distributions or skin friction are obtained, they are also reported in the form of tables or graphs at particular values of x. It seems natural therefore to construct an orthogonal coordinate system on the ship hull in which x = constant form one family of coordinate lines. This will greatly facilitate the use of the hull specifications and the results of the potential flow calculations or experimental pressure distributions in the solution of the boundary layer equations and also make the comparison with available data easier. Once such a choice is made, we need to determine the orthogonal trajectories to the x = constant curves and then find the curvatures of both families of curves. In what follows we shall denote the orthogonal trajectories by η = constant.

Let the equation of an arbitrary surface be given by

$$F(x,y,z) = 0 . \tag{28}$$

Two curves denoted by ξ = constant and η = constant are prescribed on this surface. These may be represented by the parametric relations

$$\begin{aligned} \xi &= \text{constant:} \quad x = x_1(t), \quad y = y_1(t), \quad z = z_1(t) \\ \eta &= \text{constant:} \quad x = x_2(t), \quad y = y_2(t), \quad z = z_2(t) , \end{aligned} \tag{29}$$

t being an arbitrary parameter. The condition for the two curves to be orthogonal on the surface is

$$\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 + \dot{z}_1\dot{z}_2 = 0, \tag{30}$$

where the dot represents differentiation with respect to t.

If we now align the curve ξ = constant with x = constant, as suggested earlier, then equation (30) reduces to

$$\dot{y}_1\dot{y}_2 + \dot{z}_1\dot{z}_2 = 0 . \tag{31}$$

From equation (28) it follows that

$$F_y\dot{y}_1 + F_z\dot{z}_1 = 0 . \tag{32}$$

Substitution of equation (32) into equation (31) then gives

$$\dot{z}_2 = \frac{F_z}{F_y}\dot{y}_2 , \tag{33}$$

which is a differential equation for z_2, and therefore also for x_2. It is convenient to choose x as a parameter for the family η = constant. Hence, equation (33) may be written

$$\frac{dz}{dx} = \frac{F_z}{F_y}\frac{dy}{dx} , \tag{34}$$

where the subscript 2 has been dropped for clarity. Equation (34) relates the functions z(x) and y(x), and hence determines the equation of the curve η = constant.

For the particular case of ship hulls, we choose to represent the surface by equation (1) rather than by equation (28), so that

$$F(x,y,z) = y - f(x,z) = 0 . \tag{35}$$

The derivative of y with respect to x is then

$$\frac{dy}{dx} = f_x + f_z \frac{dz}{dx} . \tag{36}$$

Substitution of this in equation (34) then gives

$$\frac{dz}{dx} = -\frac{f_x f_z}{1 + f_z^2} , \tag{37}$$

the solution of which is the required equation for $\eta(x,z)$ = constant. Thus, when f(x,z) is prescribed, it is a simple matter to find the orthogonal trajectories to the family of lines x = constant. For example, for bodies of revolution given by

$$y^2 + z^2 = g^2(x) \tag{38}$$

equation (37) gives

$$\eta = \frac{z}{g(x)} = \text{constant} . \tag{39}$$

For an ellipsoid given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{40}$$

the solution of equation (37) is

$$\eta = z\left(1 - \frac{x^2}{a^2} - \frac{z^2}{c^2}\right)^{-\frac{b^2}{2c^2}} = \text{constant}, \tag{41}$$

which happens to be the equation of the potential flow streamlines at the surface of the ellipsoid placed in a uniform stream in the x direction.

Equation (37) has been integrated numerically using a fourth-order Runge-Kutta method for the case of a parabolic ship defined by the equation

$$y = 0.1\ (1 - x^2)\ (1 - 64z^2) \tag{42}$$

to determine the orthogonal trajectories η = constant. The projections of these on the ship centerplane are shown in Figure 1. These may be compared with the projections of the lines of principal curvature, as obtained from the solution of equation (25), shown in Figure 2.

Having determined the governing equation for the curves η = constant which are orthogonal to x = constant on a ship hull, we now proceed to find the normal and geodesic curvatures of the x and η lines, using the general relationships given in Section II.

Along the x = constant curves we have

$$\dot{x} = \ddot{x} = 0\ ;\quad \dot{z} = 1\ ,\quad \ddot{z} = 0\ ;\quad I = 1 + f_z^2\ ;\quad II = f_{zz}\ ; \tag{43}$$

where z serves as a parameter. Substituting these into equations (13) and (14) we obtain

$$K_n = -\frac{f_{zz}}{(1 + f_z^2)\ (1 + f_x^2 + f_z^2)^{\frac{1}{2}}} \tag{44}$$

and

$$K_g = \frac{f_x f_{zz}}{(1 + f_z^2)^{3/2}\ (1 + f_x^2 + f_z^2)^{\frac{1}{2}}}\ . \tag{45}$$

Similarly, along the η = constant curves we have, choosing x as a parameter,

$$\dot{x} = 1 \; , \quad \ddot{x} = 0 \; ; \quad \dot{z} = -\frac{f_x f_z}{1 + f_z^2} \; ,$$

$$\ddot{z} = -\frac{f_z}{1 + f_z^2}\left[f_{xx} - \frac{f_x f_z f_{xz}}{1 + f_z^2}\right] - \frac{f_x(1 - f_z^2)}{(1 + f_z^2)^2}\left[f_{xz} - \frac{f_x f_z f_{zz}}{1 + f_z^2}\right] \; ; \tag{46}$$

$$I = 1 + \frac{f_x^2}{1 + f_z^2} \; ; \quad II = f_{xx} - \frac{2 f_x f_z f_{xz}}{1 + f_z^2} + \frac{f_x^2 f_z^2 f_{zz}}{(1 + f_z^2)^2} \; .$$

When these are inserted in equations (13) and (14) the following expressions are obtained for the curvatures:

$$K_n = -\frac{f_{xx}(1 + f_z^2)^2 - 2 f_x f_z f_{xz}(1 + f_z^2) + f_x^2 f_z^2 f_{zz}}{(1 + f_z^2)\,(1 + f_x^2 + f_z^2)^{3/2}} \tag{47}$$

and

$$K_g = \frac{f_x[f_{xz}(1 + f_z^2) - f_x f_z f_{zz}]}{(1 + f_z^2)^{3/2}(1 + f_x^2 + f_z^2)} \; . \tag{48}$$

Equations (44), (45), (47) and (48) are the final expressions for the normal and geodesic curvatures of the orthogonal family of curves x = constant and η = constant suggested for ship boundary layers.

The metric coefficients along the x and η directions can be found from the general relations set out in Section III by recognizing that along x = constant we have

$$\xi_x = 1 \; , \quad \xi_z = 0 \; , \tag{49}$$

and along η = constant we have

$$\eta_x dx + \eta_z dz = 0$$

which, upon using equation (37), gives

$$\frac{\eta_x}{\eta_z} = -\frac{dz}{dx} = \frac{f_x f_z}{1 + f_z^2} \; . \tag{50}$$

Substitution of equations (49) and (50) in equations (21) and (22) then leads to

$$h_x = \left\{1 + \frac{f_x^2}{1 + f_z^2}\right\}^{\frac{1}{2}} \tag{51}$$

and

$$h_\eta = \frac{(1 + f_z^2)^{\frac{1}{2}}}{\eta_z} \; . \tag{52}$$

We note here that the function $\eta(x,z)$, which is the solution of equation (50), appears explicitly in the geometrical properties of the x, η coordinate system only via the metric coefficient h_η, equation (52). An examination of the boundary layer equations written in the x, η coordinate system shows that, apart from the curvature terms which can all be expressed in terms of the derivatives of f, h_η always occurs in the combination $\frac{1}{h_\eta}\frac{\partial}{\partial\eta}$ operating on the velocity component in the direction of constant x. Since we have chosen x and η as independent variables, and the partial derivative with respect to η implies constant x, we have

$$\frac{1}{h_\eta}\frac{\partial}{\partial\eta} = \frac{1}{h_\eta \eta_z}\frac{\partial}{\partial z}$$

which, from equation (52), may also be written

$$\frac{1}{h_\eta}\frac{\partial}{\partial\eta} = \frac{1}{h_z}\frac{\partial}{\partial z} \quad ,$$

where

$$h_z = (1 + f_z^2)^{\frac{1}{2}} \; .$$

This shows that, once the curvatures of the x,η lines have been found, it is no longer necessary to work with the x,η coordinates; η and h_η can be eliminated altogether from the boundary layer equations using the above transformation. Thus, for the calculation of the boundary layer on ship hulls we can use x,z and the distance normal to the hull as the independent space variables. It is to be emphasized that this does not mean that we have abandoned the orthogonal x,η network in favor of the nonorthogonal x,z network. We simply use the fact that since

$\eta(x,z)$ = constant, the derivatives with respect to η (with constant x) can be transformed to those with respect to z. The solution of the boundary layer equations in the x,z plane offers the major computational advantage that the pressure distribution can be prescribed much more readily in terms of x and z than in terms of x and η.

From the results obtained in this section we see that for the coordinate system proposed here the various geometrical properties appearing in the boundary layer equations can all be expressed in terms of the derivatives of the function f(x,z) describing the body surface. As an example, the formulae obtained here have been applied to the case of an ellipsoid given by equation (40) and the results are summarized in the Appendix. The ellipsoid is a form of particular interest for three-dimensional boundary-layer work for a number of reasons. First of all, an exact solution for the potential flow past an ellipsoid at zero incidence is known so that boundary layer calculations can be performed as soon as a coordinate system is chosen. Secondly, an exact solution for the laminar boundary layer near the forward stagnation point is also available (Miloh, 1972), so that the subsequent development of the boundary layer can be calculated using this solution as the initial condition. Thirdly, by studying a number of ellipsoids with different axis ratios it will be possible to examine, systematically, the influence of both transverse and longitudinal surface curvatures not only on the behavior of the three-dimensional boundary layer but also on the applicability of the usual thin boundary-layer approximations. Finally, some experimental measurements in turbulent boundary layers on ellipsoids are available from the work of Pavamani (1960) and Eichelbrenner and his collegues (1966) which can be used as test cases for the turbulent calculation methods.

For the practically more important case of ship hulls, the function f is usually prescribed numerically. It is therefore necessary to have available a procedure for the numerical evaluation of the derivatives f_x, f_z, f_{xx}, f_{xz}, and f_{zz}. Such a procedure is described in the next section.

VI. EVALUATION OF THE DERIVATIVES OF f USING SPLINE FUNCTIONS.

The coordinates of a ship hull are usually prescribed in the form of a family of curves. Thus, for example, we may be given the curves generated by the intersection of the hull with planes (x = constant) perpendicular to the free surface at various stations along the ship length or, alternatively, the curves generated by the intersection of the hull with planes (z = constant) parallel to the free surface at different depths. From such a family of curves we can obtain a matrix f_{ij}, the elements of which represent the values of the function f(x,z) at discrete points x_i, z_j. The problem then is to find, with sufficient accuracy, the derivatives of f with respect to x and z at an arbitrary point (x,z) on the hull. There are of course a number of well known numerical techniques that can be used for this purpose. Here we shall make use of spline functions since these appear to offer some distinct advantages. Miloh (1971) has used spline functions to calculate the direction cosines of the normal to an arbitrary three-dimensional surface in connection with the solution of potential flow about ship form.

Spline functions are a class of piecewise continuous polynomials which satisfy certain continuity requirements, and have been found to possess highly desirable characteristics as approximating, interpolating, and curve-fitting functions. Although spline functions have appeared in the literature on numerical analysis only recently, being first introduced in a paper by Schoenberg (1946), there are now a number of books which describe the rigorous mathematical development of the subject (see Schoenberg (1969), Greville (1969), and Ahlberg, Nilson and Walsh (1967)). According to Greville, "There is now considerable evidence that in many circumstances a spline function is a more adaptable approximating function than a polynomial involving a comparable number of parameters. This conclusion is based in part on actual numerical experience, and in part on mathematical demonstrations that the solutions of a variety of problems of "best" approximation actually turn out to be spline functions."

The name spline is derived from the long, thin strips of plastic used by draftsmen to fair a smooth curve through a given set of points. These strips or "splines" are anchored by heavy metal blocks, known as ducks, at specific points. By changing the number and location of the ducks the spline is made to pass through all the prescribed points. The theory of elastic beams shows that the second derivative of the deflection curve is proportional to the bending moment. Since the ducks are simple supports, the bending moments, and hence the second derivative of the deflection curve, vary linearly with the distance between two neighboring supports. This is essentially the basis for the mathematical model of the so-called cubic spline, in which a piecewise, second-order polynomial is used to describe a function between two discrete points.

For a given set of values f_0, f_1, $f_2 \ldots\ldots\ldots f_n$ of the function f, associated with some constant x_i and $a = z_0 < z_1 < z_2 \ldots\ldots\ldots < z_n = b$, we now seek n cubic polynomials, one for each sub-interval $z_{j-1} < z < z_j$, such that the polynomials, together with their first and second derivatives are continuous at the given z points. Following the notation of Ahlberg, Nilson and Walsh (1967), we denote the spline function on f in the general interval $z_{j-1} < z < z_j$ by $S_f(z)$. The derivatives of the spline function will be indicated by primes. The values of the first and second derivatives of $S_f(z)$ at the end points of the interval will be denoted by m and M, respectively, with an appropriate subscript. Thus, for example, $S_f'(z_j) = m_j$ and $S_f''(z_j) = M_j$. The length of the interval in z will be denoted by h_j, i.e. $h_j = z_j - z_{j-1}$. Then, from the linearity of the second derivative of the spline function, we have

$$S_f''(z) = M_{j-1} \frac{z_j - z}{h_j} + M_j \frac{z - z_{j-1}}{h_j} . \tag{53}$$

If we integrate this twice and evaluate the constants of integration by using the conditions at z_j and z_{j-1}, we obtain

$$S_f'(z) = - M_{j-1} \frac{(z_j - z)^2}{2h_j} + M_j \frac{(z - z_{j-1})^2}{2h_j} + \frac{f_j - f_{j-1}}{h_j} - \frac{M_j - M_{j-1}}{6} h_j , \tag{54}$$

$$S_f(z) = M_{j-1} \frac{(z_j - z)^3}{6h_j} + M_j \frac{(z - z_{j-1})^3}{6h_j} + (f_{j-1} - \frac{M_{j-1} h_j^2}{6}) \frac{z_j - z}{h_j}$$

$$+ (f_j - \frac{M_j h_j^2}{6}) \frac{z - z_{j-1}}{h_j} . \qquad (55)$$

For the present case, and many other applications, it is more convenient to work with the first derivatives m_j rather than the second derivatives. In this case equations (53), (54) and (55) may be written in the alternative form:

$$S''_f(z) = -2m_{j-1} \frac{2z_j + z_{j-1} - 3z}{h_j^2} - 2m_j \frac{2z_{j-1} + z_j - 3z}{h_j^2}$$

$$+ 6 \frac{f_j - f_{j-1}}{h_j^3} (z_j + z_{j-1} - 2z) , \qquad (56)$$

$$S'_f(z) = m_{j-1} \frac{(z_j - z)(2z_{j-1} + z_j - 3z)}{h_j^2} - m_j \frac{(z - z_{j-1})(2z_j + z_{j-1} - 3z)}{h_j^2}$$

$$+ \frac{f_j - f_{j-1}}{h_j^3} 6 (z_j - z)(z - z_{j-1}) , \qquad (57)$$

$$S_f(z) = m_{j-1} \frac{(z_j - z)^2 (z - z_{j-1})}{h_j^2} - m_j \frac{(z - z_{j-1})^2 (z_j - z)}{h_j^2}$$

$$+ f_{j-1} \frac{(z_j - z)^2 [2(z - z_{j-1}) + h_j]}{h_j^3} + f_j \frac{(z - z_{j-1})^2 [2(z_j - z) + h_j]}{h_j^3} . \qquad (58)$$

If the condition of continuity of $S''_f(z)$ is now imposed at the interior points z_j $(0<j<N)$, equations (53) and (56) lead to the following two sets of linear algebraic equations in M_j and m_j:

$$\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = \frac{6}{h_j + h_{j+1}} \left\{ \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{f_j - f_{j-1}}{h_j} \right\} , \tag{59}$$

$$\lambda_j m_{j-1} + 2m_j + \mu_j m_{j+1} = 3\lambda_j \frac{f_j - f_{j-1}}{h_j} + 3\mu_j \frac{f_{j+1} - f_j}{h_{j+1}} , \tag{60}$$

where

$$\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}} , \quad \mu_j = 1 - \lambda_j , \quad (j = 1,2,3.....N-1). \tag{61}$$

Both sets are indeterminate since there are only N-1 equations for N+1 unknown in each. It is of course necessary to solve only one set of equations since the spline function is given equally well by equation (55) if M_j are found and by equation (58) if m_j are found. In order to make these determinate, we need to specify two boundary conditions at the points z_0 and z_N. The simplest choice may again be borrowed from the draftsman's spline, where the bending moment, and therefore the second derivative (curvature), is zero at the ends of the spline. This will make equations (59) determinate since $M_0 = M_N = 0$. As pointed out by Bialek and Bernicker (1969), more general end conditions, involving some linear combination of the values of the function and its first and second derivatives at z_0 and z_N, can also be used. Here, we choose to make the second set, equations (60), determinate by assuming that the first derivatives m_0 and m_N are known, since these can easily be found from the given ship data. We may write these equations in matrix notation in the form

$$[A_{ij}]\,[m_j] = [C_j] , \tag{62}$$

where m_j and C_j are column matrices with N-1 elements while A_{ij} is a square tri-diagonal matrix. Equation (62), with m_0 and m_N known, can be solved for m_j using standard techniques. A particularly efficient method for solving this type of equations, used in the numerical example mentioned later on, is the Thomas algorithm (Ames, 1965). Once m_j have been determined, equations (57), (58) and (56) can be used to find, respectively, the values of the function and its first and second derivatives at any desired point in $z_0 \leq z \leq z_N$.

For the ship hull, which, as mentioned earlier, is prescribed by the matrix f_{ij}, we can apply the general procedure described above to determine f_x, f_{xx}, f_z and f_{zz} by generating the spline functions $S_f(x)$ and $S_f(z)$ on f in the x and z directions, respectively. Then $f_x = S_f'(x)$, $f_{xx} = S_f''(x)$, $f_z = S_f'(z)$ and $f_{zz} = S_f''(z)$. In order to find the cross-derivative f_{xz} we can obtain the spline function $S_{f_x}(z)$ on f_x in the z direction, or the function $S_{f_z}(x)$ on f_z in the x direction, so that $f_{xz} = S_{f_x}'(z) = S_{f_z}'(x)$. It can be shown quite rigorously that, for a rectangular domain where f_{xz} is specified at the four corners, the values of f_{xz} determined from the two splines are identical.

The method of splines described here has been applied to the case of the parabolic ship, given by equation (42), and the values of the various derivatives calculated in this manner are compared with the exact values in Table 1.

VII. DISCUSSION

A number of methods are now available for the calculation of three-dimensional boundary layers. Some of the better known ones are listed in Table 2. The older integral methods for laminar and turbulent boundary layers are described in a review article by Cooke and Hall (1962), while more recent developments in the theory of three-dimensional turbulent boundary layers have been considered in a book by Nash and Patel (1972). As we have already remarked, the basic assumptions in the integral methods are such that they are wedded to the use of streamline coordinates. Their use in the treatment of boundary layers on arbitrary surfaces therefore requires the prior knowledge of the streamlines in the potential flow. In methods which solve the differential equations of the boundary layer by numerical techniques, on the other hand, the equations are formulated in coordinate systems which are less restrictive, the only stipulation being that the coordinates be orthogonal on the surface. Upto the present time, however, such methods have been applied to a limited number of experimentally observed three-dimensional flows and, with the exception of the methods of Raetz (1957) and Nash and Patel (1971), all use rectangular Cartesian coordinates. In the work of Nash and Patel, a number of flows were calculated using the boundary layer equations written in non-Cartesian, curvilinear coordinates. Although the choice of a convenient coordinate system became apparent for each case, the curvatures and the metric coefficients associated with each set of coordinates had to be evaluated on an ad hoc basis. If the differential methods of the type now being proposed are to be applied to perform boundary layer calculations for more arbitrary surfaces, such as ship forms, it is of course necessary to develop more general procedures for the practical evaluation of coordinate curvatures and metric coefficients. The work described here is an attempt to meet this immediate need.

The results on the curvatures and metric coefficients given in Sections II and III apply to any coordinate system that is locally orthogonal on the surface, including of course the streamline coordinate system. The method of obtaining the lines of principal curvature, and the principal curvatures, of a surface, described in Section IV, serves two useful purposes. First, it enables one to construct the unique triply-orthogonal system which has to be used whenever the thin boundary-layer approximations do not apply. Secondly, a comparison of the boundary layer thickness δ, calculated using the thin boundary-layer equations, with the principal curvatures K_n of the surface will indicate whether the usual assumption $\delta K_n \ll 1$ needs to be re-examined in some region of the flow. Such a situation is most likely to occur in the neighborhood of ship sterns. For thin boundary layers, however, the use of the lines of principal curvature to form the coordinate system is not very convenient since the determination of these is rather complicated. Moreover, as we have seen in the case of a parabolic ship, the curvatures of these lines may be quite large on some parts of the hull so that their numerical evaluation with sufficient accuracy requires greater care.

The coordinate system proposed in Section V for the calculation of three-dimensional boundary layers on ship hulls appears to be the most convenient one for the practical reasons already discussed. The detailed results for this particular coordinate system indicate that it is also a convenient system from a numerical point of view since the equations of the coordinate lines, x = constant and η = constant, as well as the curvatures and the metric coefficients associated with these lines, can all be expressed in terms of the first and second derivatives of the function describing the ship hull, which, in turn, can be found using simple spline functions. Consideration of the boundary layer equations further suggests that they can be transformed from the orthogonal x,η coordinates to those in the non-orthogonal x,z coordinates, drawn on the hull, without loss of generality. The equations written in the latter system offer additional computational advantages.

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APPENDIX: Application of the Proposed Coordinate System to an Ellipsoid.

For an ellipsoid given by equation (40) we have

$$y = f(x,z) = b\left(1 - \frac{x^2}{a^2} - \frac{z^2}{c^2}\right)^{\frac{1}{2}} \quad , \quad y \geq 0 \; ,$$

and, from equation (41), the projections of the η = constant lines on the y = 0 plane are given by

$$\eta(x,z) = z\left(1 - \frac{x^2}{a^2} - \frac{z^2}{c^2}\right)^{-\frac{b^2}{2c^2}} .$$

Introduction of these in equations (44), (45), (47), (48), (51) and (52) leads to the following results:

$$(K_n)_{x = \text{constant}} = \frac{b}{c^2} \frac{1 - \bar{x}^2}{AB^{\frac{1}{2}}} \quad ,$$

$$(K_g)_{x = \text{constant}} = \frac{b^2}{ac^2} \frac{\bar{x}(1 - \bar{x}^2)}{A^{3/2} \; B^{\frac{1}{2}}} \quad ,$$

$$(K_n)_{\eta = \text{constant}} = \frac{b}{a^2} \frac{(1 - \bar{z}^2)A^2 - \frac{b^2}{c^2} \bar{x}^2 \bar{z}^2 \left[2A - \frac{b^2}{c^2}(1 - \bar{x}^2)\right]}{AB^{3/2} \; (1 - \bar{x}^2 - \bar{z}^2)} \quad ,$$

$$(K_g)_{\eta = \text{constant}} = \frac{b^2}{a^2 c} \left(1 - \frac{b^2}{a^2}\right) \frac{\bar{x}^2 \bar{z}(1 - \bar{x}^2 - \bar{z}^2)^{\frac{1}{2}}}{A^{3/2} \; B} \quad ,$$

$$h_x = \left\{\frac{1 + \frac{b^2}{a^2} \bar{x}^2}{A}\right\}^{\frac{1}{2}} \quad ,$$

and

$$h_z = \left\{\frac{A}{(1 - \bar{x}^2 - \bar{z}^2)}\right\}^{\frac{1}{2}} \quad ,$$

where

$$\bar{x} = \frac{x}{a} \; , \; \bar{z} = \frac{z}{c} \quad ,$$

$$A = 1 - \bar{x}^2 - \bar{z}^2\left(1 - \frac{b^2}{c^2}\right) \quad ,$$

and

$$B = A + \frac{b^2}{a^2} \bar{x}^2 \; .$$

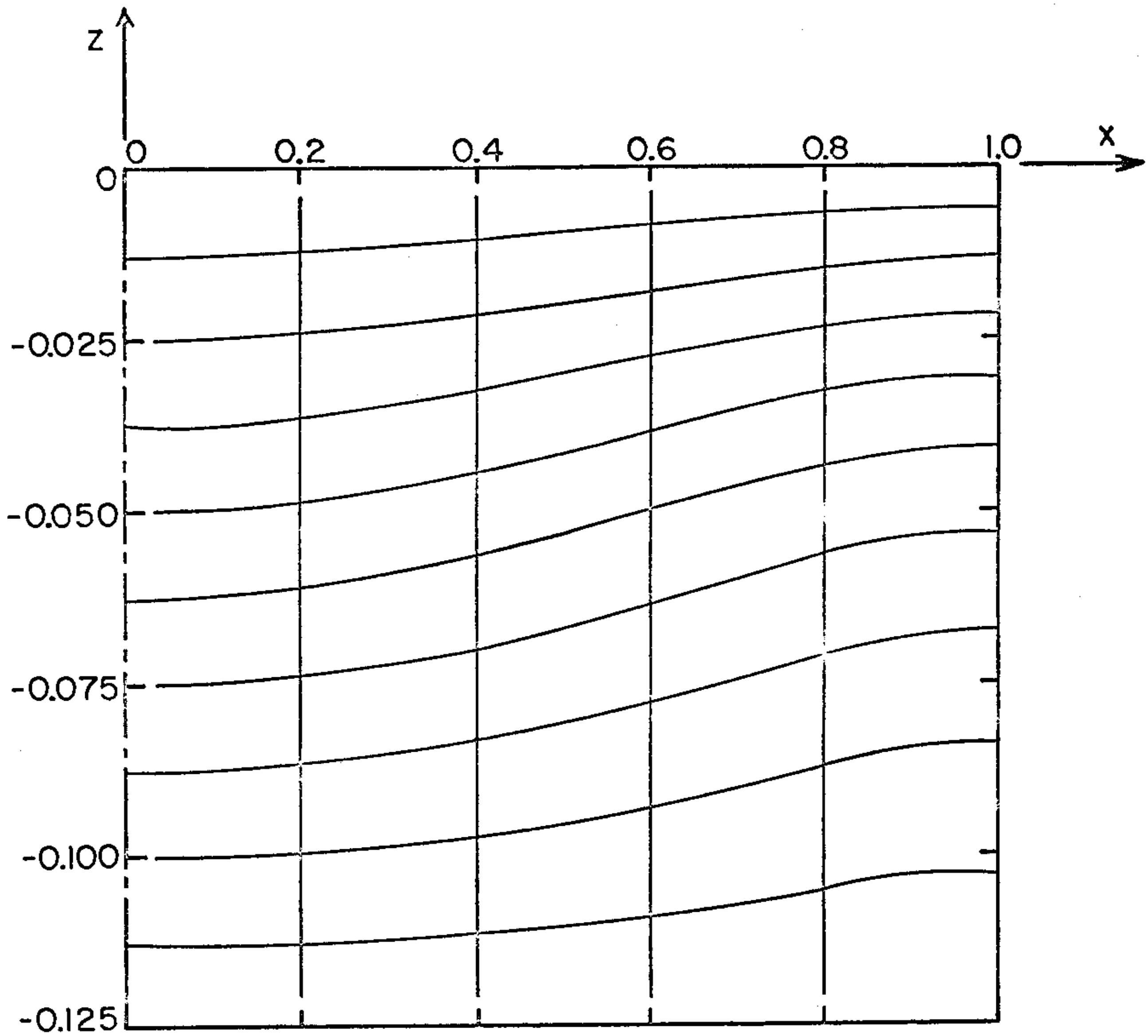

Figure 1. Projections of the Proposed Coordinates on the Centerplane for a Parabolic Ship (Equation 42). (Note the 8:1 contraction in x)

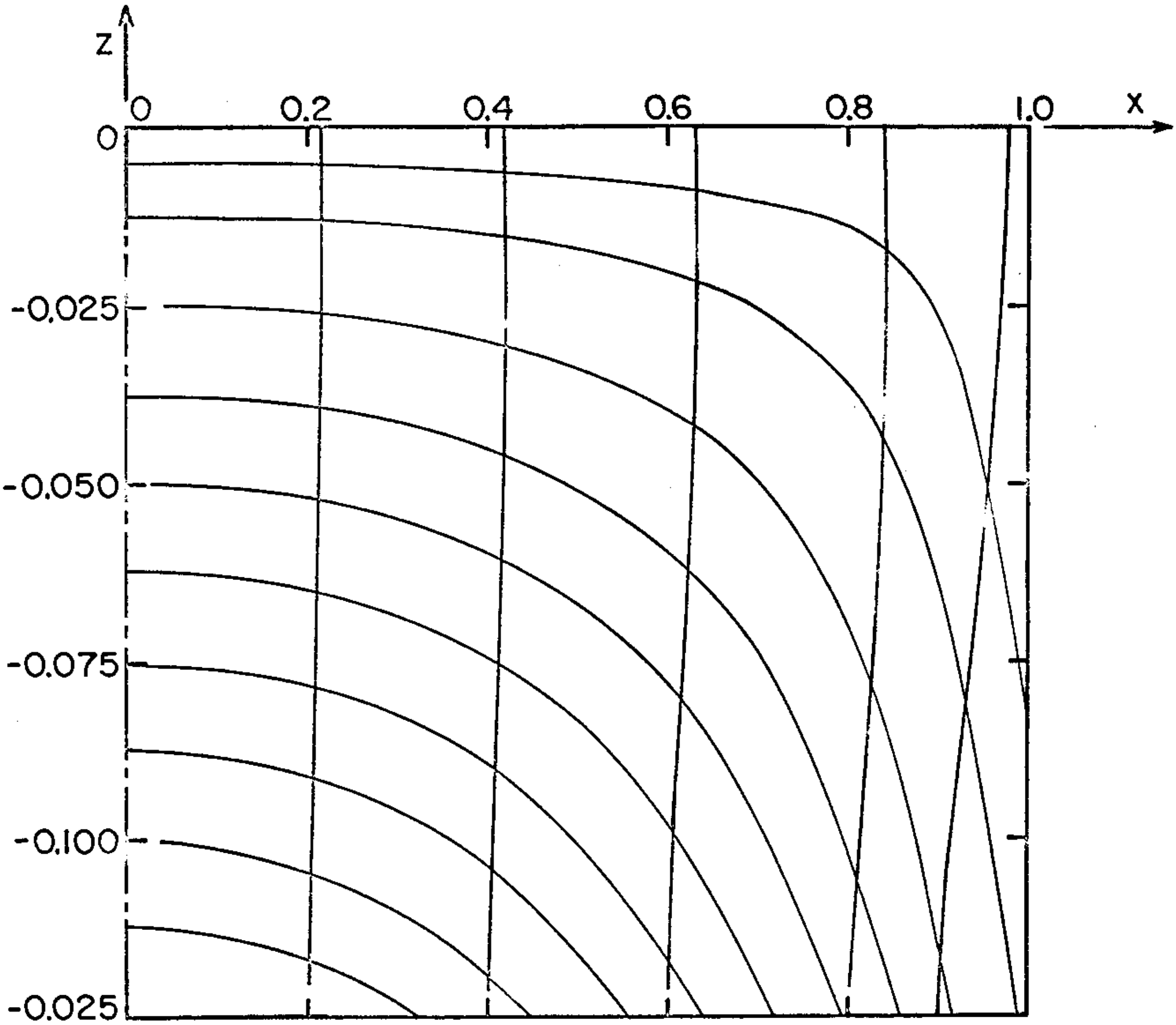

Figure 2. Projections of the Lines of Principal Curvature of a Parabolic Ship (Equation 42) on the Centerplane. (Note the 8:1 contraction in x)

x	z	f_x	f_z	f_{xx}	f_{zz}	f_{xz}
.9000	-.0125	-.178200	.030400	-.198011	-2.431969	- .288000
		-.178200*	.030400*	-.198000*	-2.431999*	- .288000*
.9000	-.0625	-.135000	.152000	-.150003	-2.432007	-1.440000
		-.135000*	.152000*	-.150000*	-2.431999*	-1.439999*
.9000	-.1125	-.034200	.273599	-.038001	-2.431931	-2.591988
		-.034200*	.273600*	-.038000*	-2.431999*	-2.591998*
.7000	-.0375	-.127400	.244800	-.182014	-6.527893	- .672004
		-.127400*	.244800*	-.182000*	-6.527998*	- .672000
.7000	-.0875	-.071400	.571199	-.102012	-6.527878	-1.567997
		-.071400*	.571199*	-.102000*	-6.527998*	-1.567999*
.5000	-.0125	-.099001	.120001	-.198023	-9.600006	- .160012
		-.099000*	.120000*	-.198000*	-9.599998*	- .160000*
.5000	-.0625	-.075000	.599998	-.150017	-9.600342	- .799999
		-.075000*	.600000*	-.150000*	-9.599998*	- .800000*
.5000	-.1125	-.019000	1.079998	-.038002	-9.599608	-1.440001
		-.019000*	1.079998*	-.038000*	-9.599998*	-1.440000*
.3000	-.0375	-.054599	.436804	-.181999	-11.647330	- .288008
		-.054600*	.436800*	-.182000*	-11.647990*	- .288000*
.3000	-.0875	-.030600	1.019198	-.101999	-11.647210	- .671992
		-.030600*	1.019198*	-.102000*	-11.647990*	- .672000*
.1000	-.0125	-.019801	.158401	-.198002	-12.671170	- .031993
		-.019800*	.158400*	-.198000*	-12.671990*	- .032000*
.1000	-.0625	-.015000	.792000	-.150001	-12.671870	- .159990
		-.015000*	.792000*	-.150000*	-12.671990*	- .160000*
.1000	-.1125	-.003800	1.425597	-.037998	-12.671630	- .288013
		-.003800*	1.425598*	-.038000*	-12.671990*	- .288001*

Table 1. Derivatives for the Parabolic Ship. Comparison between exact values (denoted by *) and those found using cubic splines with 20 points in x- and 10 points in z-direction.

Method proposed by	Classification	Application	Coordinates
Timman (1950)	Integral	Laminar	Streamline
Zaat (1956)	Integral	Laminar	Streamline
Raetz (1957)	Differential	Laminar	Curvilinear Orthogonal
Mager (1951)	Integral	Turbulent	Streamline
Cooke (1958)	Integral	Turbulent	Streamline
Eichelbrenner & Peube (1966)	Integral	Turbulent	Streamline
Smith (1966)	Integral	Turbulent	Streamline
Cumpsty & Head (1967)	Integral	Turbulent	Streamline
Nash (1969)	Differential	Turbulent	Cartesian
Bradshaw (1971)	Differential	Turbulent	Cartesian
Nash & Patel (1971)	Differential	Turbulent	General Curvilinear Orthogonal
Pierce & Klinksiek (1971)	Differential	Turbulent & Laminar	Cartesian

Table 2. Some of the Methods Available for the Calculation of Three-Dimensional Boundary Layers

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13. ABSTRACT

The problem of choosing orthogonal, curvilinear, coordinate systems for use in boundary-layer calculations on arbitrary three-dimensional bodies is considered in some detail. A general method for the practical evaluation of the various geometrical properties of the coordinates occurring in the three-dimensional boundary-layer equations is described. A particular coordinate system which appears to be the most convenient one for ship hulls is then proposed and analyzed further.

Unclassified

Security Classification

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Ship forms						
Coordinate systems						
Lines of principal curvature						
Spline functions						

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