SIMILARITY SOLUTIONS FOR PARTIAL DIFFERENTIAL EQUATIONS GENERATED BY FINITE AND INFINITESIMAL GROUPS

by

Henry S. Woodard and William F. Ames

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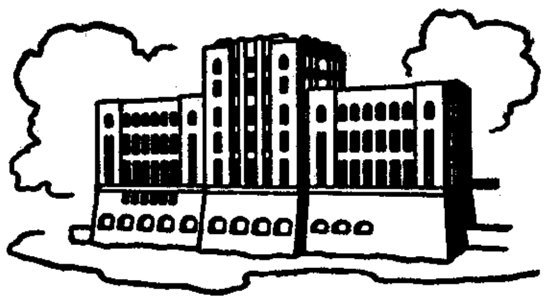

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Iowa Institute of Hydraulic Research
The University of Iowa
Iowa City, Iowa

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ABSTRACT

The problem of developing systematic methods for obtaining similarity variables is considered for partial differential equations. Similarity variables are a set of transformations which reduce a partial differential equation to an ordinary differential equation.

This paper considers two methods of generating similarity variables. The first method uses a group of finite transformations and the second uses a group of infinitesimal transformations. The mathematical theory for both techniques is described and illustrated.

The two methods of obtaining similarity variables are applied to the Burgers' equation $u_y + uu_x = u_{xx}$ and to the laminar boundary layer equations with a pressure gradient. In all cases considered, new types of similarity variables are found. In addition, the auxiliary conditions are discussed in the light of the new similarity variables obtained for the boundary layer equations.

TABLE OF CONTENTS

Chapter		Page
	LIST OF FIGURES	v
	LIST OF SYMBOLS	vi
I	INTRODUCTION	1
II	SIMILARITY BY FINITE GROUP METHODS	9
	Continuous Groups	9
	Invariant Transformations	11
	Absolute Invariants	14
	Similarity	16
	Boundary Conditions	20
III	SIMILARITY BY INFINITESIMAL GROUPS	22
	Infinitesimal Transformations	22
	Invariants	25
	Canonical Variables	26
	Reduction of the Number of Independent Variables in a Partial Differential Equation	28
	Similarity Solutions in Practice	31
IV	APPLICATION OF THE FINITE GROUP METHOD	35
	Burgers' Equation	35
	Laminar Boundary Layer	42

TABLE OF CONTENTS CONT'D.

Chapter		Page
V	APPLICATION OF THE INFINITESIMAL GROUP METHOD. . . .	61
	Burgers' Equation	61
	Boundary Layer Equations	74
VI	CONCLUSION	88
	BIBLIOGRAPHY	91

LIST OF FIGURES

Figure		Page
1.1	Heat Conduction in Circular Cylinders	2
1.2	Profile for Velocity in Boundary Layer	85

LIST OF SYMBOLS

ϵ	= infinitesimal parameter
a_i, a	= finite continuous parameter
U_∞	= free stream velocity
ν	= kinematic viscosity
U_c	= characteristic velocity
u^*	= component of velocity in the x-direction
v^*	= component of velocity in the y-direction
η, η_i, Ω, ζ, λ_i, g_i	= absolute invariants
f_i	= functions which define a finite transformation
X_i, Y, U, V, X	= functions which define an infinitesimal transformation.
ϕ	= a differential form
$\bar{V}$, Q	= differential operators
I	= invariant solutions
$J_{1/3}$	= Bessel function of the first kind
$Y_{1/3}$	= Bessel function of the second kind
P	= unknown function in the boundary layer equations
J	= Jacobian
a^o	= identity element

CHAPTER I

INTRODUCTION

This manuscript concerns itself with the problem of developing systematic methods for finding new similarity variables for partial differential equations in engineering. Similarity variables are a set of transformations which, when applied, reduce the number of independent variables in a partial differential equation. The advantage of such a transformation is the simplification of the problem at hand.

Similarity is often associated with the physical nature of the problem. For example, steady state heat conduction in a hollow circular cylinder is described by Laplace's equation $T_{xx} + T_{yy} = 0$, where T is the temperature and x and y are cartesian coordinates. If the temperature is constant on the inner and outer surface (see Fig. 1.1). the physical symmetry suggests that the isotherms are concentric circular cylinders and the temperature distribution could be described by a single variable whose constants corresponded to the radii of the circular isotherms. Hence, we may postulate from a physical standpoint that a reduction in the number of independent variables is plausible and a transformation such as

$$\begin{aligned} &T(x, y) \rightarrow T(R) \\ &R = (x^2 + y^2)^{\frac{1}{2}} \end{aligned} \qquad (1.1)$$

is appropriate. The transformation (1.1) is a set of similarity

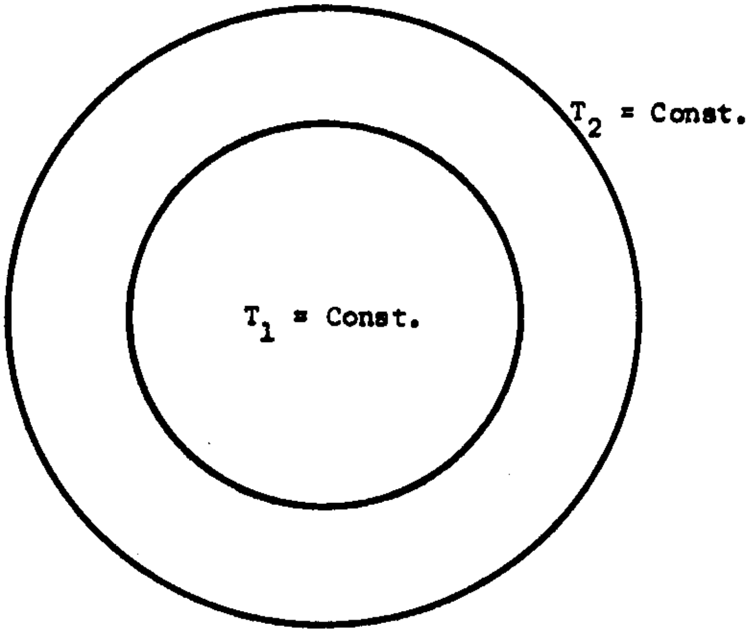

Fig. 1.1 Heat Conduction in Circular Cylinders

variables for Laplace's equation.

Boltzmann[1]* considered the problem of nonlinear diffusion which is expressed by the equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} [D(T) \frac{\partial T}{\partial x}] \tag{1.2}$$

and found the similarity variables

$$\eta = xt^{-\frac{1}{2}}$$
$$T(x, y) \rightarrow T(\eta). \tag{1.3}$$

Later Blasius [2] postulated the form of similarity variables by reasoningabout the physical nature of the problem. Although this method worked adequately for some problems, many other problems were too complicated to attack without a more systematic approach. One answer to this shortcoming was a method which will be referred to as the separation of variables. As an illustration, suppose we have a partial differential equation with one dependent variable T and two independent variables x and y. Separation of variables entails searching for a transformation of the form

$$\eta = y\, g\,(x)$$
$$T = h(x, y)\, F(\eta) \tag{1.4}$$

such that the resulting application yields an ordinary differential equation in F and η. In practice, severe restrictions of the functional form of g(x) and h(x, y) are often necessitated and many writers refer

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*A bracket will denote references given in the bibliography.

to such restrictions as the necessary conditions for the existence of similarity variables.

Though restrictive, the method of separation of variables proved to be successful for many problems in boundary layer theory. Herzig and Hansen [3] considered three dimensional flows over a flat plate and assumed that the leading edge was under main flow streamlines which are translates and representable by polynomial expressions. In a later paper Herzig and Hansen [4] used separation of variables to attack three -dimensional boundary layer flows in polar coordinates and found that similarity could be achieved by restricting the functional form of the free stream velocity. Cohen and Reshotko [5], when considering compressible boundary layer flow with heat transfer and pressure gradients, found that the necessary condition for the existence of similarity placed restrictions on the combination of Prandtl number and external Mach number together with requirements on the free stream velocity. A more general attack on the boundary layer problem in curvilinear coordinates given by Hansen [6] revealed that again similarity using separation of variables placed restrictions on the free stream velocity for various curvilinear systems. These and other examples show that the method of separation of variables is effective; but because of the nature of applying the method, definite restrictions on the generality of the similarity variables occur.

Chapter II of this report describes a group theoretic technique for generating similarity variables. The groups which will be referred to here are continuous transformations with one or more parameters and have the following form:

$$\bar{x} = f_1\ (x,\ y,\ a)$$
$$\bar{y} = f_2\ (x,\ y,\ a) \qquad (1.5)$$

In order that the above transformation be a group, three properties must be satisfied. First, if composition* is the operator of the group and distinct but arbitrary values of the parameter a determine the members of the group, then the composite is also a member of the group. This requirement is called the closure property. Second, the inverse of each transformation must exist and be a member of the group. The existence of an inverse (see [7]) is guaranteed if

i) $f_1(x,\ y,\ a)$ and $f_2(x,\ y,\ a)$ are continuously differentiable at every point.

ii) the Jacobian $J = \partial(\bar{x},\ \bar{y})/\partial(x,\ y)$ is nonzero at every point.

Third, there must exist an identity element a^o such that $f_1(x,y,a^o) = x$ and $f_2(x,y,a^o) = y$.

The general theory of group techniques is placed on firm mathematical ground by Morgan [8] and Michal [9]. Morgan [10] gives a discussion of the application of his theory to the problem considered by Hansen [6] and finds that a simple group† essentially produces the same results as the method of separation of variables. For problems in engineering, detailed descriptions of the application of group techniques are given by Birkhoff [11], Hansen [12], and Ames [13]. Application of

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*The composition of two transformations is the successive performance of them.

†A group of continuous transformations with one parameter will be called simple if it has the form: $\bar{x}_i = f_i(a)\ x + g_i\ (a)$.

the group method for non-Newtonian boundary layer flows may be found in [14] and [15].

Perhaps the first effort to find and use a more general group was given by Krzywoblocki and Roth [16], [17] and [18]. For Laplace's equation in two dimensions they found the rotation group was applicable:

$$\bar{x} = x \cos a - y \sin a$$
$$\bar{y} = x \sin a + y \cos a \qquad (1.6)$$
$$\bar{u} = u.$$

Because Laplace's equation has the property that $\nabla^2 \bar{u} = w(x,y,a) \nabla^2 u$, it will be called <u>conformally invariant</u>. As will be seen later, conformal invariance is an essential property of the group technique. Krzywoblocki and Roth considered the Laplace's equation in higher dimensions and some problems in viscous fluid mechanics, but only found simple transformation groups for the latter. Their failure to find new similarity variables for fluid problems was probably due to the restrictive form of their transformation group.

The works of Morgan and Michal lacked a formal mathematical procedure to find the <u>invariants</u> of a group. A function is an invariant of a group if it has the same form in the transformed variables as in the untransformed variables. For example, in the rotation group $x^2 + y^2$ is an invariant of the group because $\bar{x}^2 + \bar{y}^2 = x^2 + y^2$. Gaggioli and Moran ([19] through [23]) found a theorem in Cohen [24] for the necessary and sufficient conditions for the existence of invariants of a group. Combining this theorem with the theory of Morgan and Michal, Gaggioli and Moran considered the problem of finding similarity variables with a general class of finite groups, but in their illustrations only simple groups were used. In addition these authors considered two parameter groups of the type

$$\bar{x}_i = C^{x_i}(a, b)\, x_i + k^{x_i}(a, b)$$

and applied the result to a three-dimensional boundary layer problem. Finally, Gaggioli and Moran developed a method for handling auxiliary conditions and demonstrated it with an illustration.

Chapter III of this report describes a relatively new method which will be referred to as the infinitesimal group method. An infinitesimal group of transformations assumes the form

$$\bar{x}_i = x_i + \varepsilon X_i (x_1, \ldots, x_n)$$

where ε is an infinitesimal parameter. Infinitesimal groups were originated by Lie [25] in order to unify the methods of solution for ordinary differential equations. In addition, Lie developed a theory of one parameter groups including both the finite and infinitesimal cases. Some authors refer to one parameter groups of transformations as Lie groups. A discussion of the application of Lie groups to ordinary differential equations and partial differential equations of the first order is given by Cohen [24]. Ovsjannikov [26] was perhaps the first one to use infinitesimal groups to generate similarity variables of partial differential equations; he considered the non-linear diffusion problem. Later Cole and Bluman [27] published a description of the infinitesimal group method and an application to the linear heat conduction equation. Bluman [28] gives a more complete discussion of the infinitesimal method and several examples.

Chapters IV and V of this report are devoted to expansion of both the theoretic group technique and the infinitesimal group method and to the application of these developments to a number of cases. The main contribution of this work is threefold. First, expanding upon the ideas of Krzywoblocki and Roth, a practical method for obtaining generalized groups is presented together with the development of appropriate similarity variables. In addition a systematic method for carrying an unknown function through the extended analysis is presented together with a procedure for determining restrictions on the nature of that function so that similarity is preserved. Boundary conditions are examined in the light of a set of necessary and sufficient conditions to determine whether they are compatible with the generalized group. Second, following the ideas suggested at the end of Cole and Bluman [27], this report uses the infinitesimal group to generate nonclassical similarity variables and extends the method to handle simultaneous partial differential equations. The treatment of the boundary conditions and unknown functions is illustrated. A physical interpretation of some new similarity variables is presented and some numerical results for problems using such variables are given. Third, a possible pitfall in the method of infinitesimal groups is noted with an explanation of its occurrence and supporting examples are presented.

CHAPTER II

SIMILARITY BY FINITE GROUP METHODS

Continuous Groups

In Chapter I continuous transformation groups have been defined. Because the properties of such groups are extremely important for the discussion which follows, they will be reviewed here in greater detail.

An r parameter continuous group of transformations has the form

$$\bar{x}_i = f_i(x_1,\ldots, x_n; a_1,\ldots, a_r), \quad i = 1,\ldots,n.$$

It will be assumed that the f_i are continuously differentiable functions of the variables x_i, $i = 1,\ldots,n$, and of the parameters a_j, $j = 1,\ldots,r$. Further, the parameters a_j must be essential in the sense that a small variation in each of them produces a definite change in the functions f_i. For example, suppose that each parameter a_j is incremented by an arbitrary small quantity ϵ_j; then the parameters are not essential if

$$f_i\,(x_1,\ldots, x_n; a_1,\ldots a_r) = f_i\,(x_1,\ldots, x_n; a_1 + \epsilon_1,\ldots, a_r + \epsilon_r).$$

An example of a group with a non-essential parameter is

$$\bar{x} = x + (a_1 + a_2)$$

$$\bar{y} = y + a_1.$$

Obviously, a_1 and a_2 can be incremented such that $a_1 + a_2$ remains unchanged. A necessary and sufficient condition (see Eisenhart [29]) for r parameters to be essential is that the functions f_i do not satisfy an equation of the form

$$\sum_{k=1}^{r} \chi_k (a_1, \ldots, a_r) \frac{\partial f_i}{\partial a_k} = 0.$$

In addition we require that the inverse set of transformations exist and that we can find parameters $\bar{a}_j$, which are functions of the parameters a_j, such that

$$x_i = f_i^{-1} (\bar{x}_1, \ldots, \bar{x}_n; \bar{a}_1, \ldots, \bar{a}_r)$$

and that

$$\bar{\bar{x}}_i = f_i(\bar{x}_1, \ldots, \bar{x}_n; \bar{a}_1, \ldots, \bar{a}_r) = f_i(f_1, \ldots, f_n; \bar{a}_1, \ldots, \bar{a}_r) = x.$$

The inverse property is guaranteed if the Jacobian J

$$J = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$$

is not zero at any point.

The closure property as stated in the introduction requires that if

$$\bar{x}_i = f_i(x_1, \ldots, x_n; a_1, \ldots, a_r)$$

$$\bar{\bar{x}}_i = f_i(\bar{x}_1, \ldots, \bar{x}_n; b_1, \ldots, b_r)$$

then

$$\bar{\bar{x}}_i = f_i(x_1, \ldots x_n; c_1, \ldots c_r)$$

where the parameters c_j are functions of the parameters a_j and b_j. That is

$$c_j = \psi_j(a_1, \ldots, a_r; b_1, \ldots, b_r).$$

Next, we require that there exists a set of parameters a_j° called the identity element such that

$$\bar{x}_i = f_i(x_1, \ldots, x_n; a_1^{\circ}, \ldots, a_r^{\circ}) = x_i.$$

Finally, the operation of composition must be associative.

Continuous r parameter groups of transformations which satisfy the inverse property, the closure property, and have an identity element are called r parameter Lie groups of transformations.

As an example consider the group

$$\bar{x} = a^{\alpha_1} x$$

$$\bar{y} = a^{\alpha_2} y.$$

The identity element is a = 1. The closure property implies that if

$$\bar{\bar{x}} = b^{\alpha_1}\, \bar{x} = (a\ b)^{\alpha_1} x$$

then there exists a $(c)^{\alpha_1} = (a\ b)^{\alpha_1}$ such that

$$\bar{\bar{x}} = (c)^{\alpha_1} x$$

belongs to the group. The inverse transformation is

$$x = a^{-\alpha_1} \bar{x}$$

$$y = a^{-\alpha_2} \bar{y}$$

Invariant Transformations

A differential form of the k^{th} order and m independent variables is denoted by

$$\phi(x_1, \ldots, x_m; y_1, \ldots, y_n; \frac{\partial^k y_1}{\partial (x_1)^k}, \ldots, \frac{\partial^k y_n}{\partial (x_m)^k}). \qquad (2.1)$$

A continuous group of transformations G_1 represented by

$$\left.\begin{array}{l} \bar{x}_i = f_i(x_1, \ldots, x_m; a) \\ \bar{y}_i = v_i(y_1, \ldots, y_n; a) \end{array}\right\} G_1 \qquad (2.2)$$

may be enlarged by appending the partial derivatives of y_i with respect to x_i up to and including the k^{th} derivative. The resulting set of transformations forms a group G_k. Such a group will be called the <u>k^{th} enlargement</u> of the group G_1. Denoting the arguments of ϕ by $z_1, \ldots, z_p$, we say that the differential form ϕ is <u>conformally invariant</u> under the enlargement G_k if

$$\phi(\bar{z}_1, \ldots, \bar{z}_p) = E(z_1, \ldots, z_p; a)\, \phi(z_1, \ldots, z_p). \qquad (2.3)$$

If E is a function of <u>a</u> only, ϕ is said to be <u>constant conformally invariant</u>; and if E is a constant, then ϕ is said to be <u>absolutely invariant</u>.

Now consider a path curve $h(x_1, x_2) = 0$ and a one parameter group

$$\bar{x}_1 = f_1(x_1, x_2, a)$$

$$\bar{x}_2 = f_2(x_1, x_2, a)$$

with the requirement that the identity element is $a^o = 0$. The curve $h(x_1, x_2) = 0$ is an <u>invariant curve</u> if $h(\bar{x}_1, \bar{x}_2) = 0$ whenever $h(x_1, x_2) = 0$. Expanding $h(\bar{x}_1, \bar{x}_2)$ in a Taylor series we obtain

$$h(\bar{x}_1, \bar{x}_2) = h(x_1, x_2) + Qha + Q^2ha^2/2 + \ldots$$

where

$$Q \equiv \frac{\partial \bar{x}_1}{\partial a}\bigg|_{a=a^o} \frac{\partial(\ )}{\partial x_1} + \frac{\partial \bar{x}_2}{\partial a}\bigg|_{a=a^o} \frac{\partial(\ )}{\partial x_2} \tag{2.4}$$

and $Q^2 \equiv QQ$. A necessary condition (see Cohen [24]) for $h(x_1, x_2)$ to be an <u>invariant curve</u> is that $Qh = 0$ whenever $h(x_1, x_2) = 0$; that is $Qh = w(x_1, x_2)\, h(x_1, x_2)$. In that case we find that $Q^2h = QQh = Qwh + wQh = (Qw + w^2)h$ and similarly if $Q^nh = \theta(x_1, x_2)h$ then $QQ^nh = (Q\theta + \theta w)h$. Hence the vanishing of Qh, whenever $h(x_1, x_2)$ does, is both a necessary and sufficient condition for $h(x_1, x_2) = 0$ to be an invariant curve. This argument extends to the differential form ϕ and leads to the following theorem (see Morgan [8]):

<u>Theorem 2.1</u>. If ϕ is at least in the class* $C^{(1)}$ with respect to each of its arguments, then a necessary and sufficient condition for ϕ to be conformally invariant under a one-parameter group of transformations is that

$$\bar{V}\phi = w_1(z_1,\ldots, z_p)\, \phi(z_1,\ldots, z_p) \tag{2.5}$$

where $\bar{V} \equiv \xi^1(z_1,\ldots, z_p) \frac{\partial}{\partial z_1}(\ ) + \ldots + \xi^p(z_1,\ldots, z_p) \frac{\partial}{\partial z_p}(\ )$

and $\xi^i = \frac{\partial f_i}{\partial a}(z_1,\ldots, z_p; a^o)$.

To give an example: the group given by equations (1.6) may be applied to Laplace's equation with the result that $\nabla^2\bar{u} = \nabla^2 u$ and we conclude that Laplace's equation is absolutely invariant with respect to that group. Suppose we consider the Poisson equation of the form

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* The class $C^{(1)}$ is the class of continuous functions with one continuous derivative.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = H(x, y, u) \tag{2.6}$$

and inquire as to the necessary conditions for this equation to be absolutely invariant under the group (1.6). In that case we have

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - H(\bar{x}, \bar{y}, \bar{u}) = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} - H(x, y, u) \tag{2.7}$$

which requires that $H(\bar{x}, \bar{y}, \bar{u}) = H(x, y, u)$. Now differentiating with respect to a and setting $a = a^o$ we see that with group (1.6)

$$-\frac{\partial H}{\partial x} y + \frac{\partial H}{\partial y} x = 0. \tag{2.8}$$

In this case the w_1 in (2.5) is zero and the above equation is the necessary and sufficient condition for $H(x, y, u)$ to be absolutely conformally invariant. The solution of (2.8) is

$$H = H_1(x^2 + y^2, u)$$

where H_1 is an arbitrary function. Of course, H_1 can be just a constant.

Absolute Invariants

Let us inquire under what condition a function $D(x, y)$ is an absolute invariant of the one parameter group

$$\bar{x} = f_1(x, y, a)$$

$$\bar{y} = f_2(x, y, a).$$

Without loss of generality, we may assume that $a^o = 0$ is the identity element. In order for $D(x, y)$ to be an absolute invariant we require that

$$D(\bar{x}, \bar{y}) = D(x, y).$$

Expanding $D(\bar{x}, \bar{y})$ in a Taylor Series we have

$$D(\bar{x}, \bar{y}) = D(x, y) + QDa + Q^2D\ a^2/2 + \ldots \qquad (2.9)$$

where $Q = \left.\frac{\partial\bar{x}}{\partial a}\right|_{a^o=o} \frac{\partial}{\partial x} + \left.\frac{\partial\bar{y}}{\partial a}\right|_{a^o=o} \frac{\partial}{\partial y}$.

Now the operator Q has the property that $Q^2D = QQD$, $Q^3D = QQ^2D, \ldots,$ $Q^n = QQ^{n-1}D$, and if $QD = 0$, then all the higher order terms in the series given by (2.9) are zero. As a result we have the theorem given by Cohen [24]:

Theorem 2.2. The necessary and sufficient condition for $D(\bar{x}, \bar{y}) = D(x, y)$ is that $QD = 0$.

The above result extends to two parameter groups. For the group $\bar{x}_i = x_i\ C_i(a_1, a_2) + K_i(a_1, a_2)$, we have the following theorem (see Gaggioli and Moran [23]):

Theorem 2.3. The necessary and sufficient conditions for η to be an invariant are

$$\sum_{i=1}^{n} \left.\frac{\partial\bar{x}^i}{\partial a_1}\right|_{\substack{a_1 = a_1^o \\ a_2 = a_2^o}} \frac{\partial}{\partial x^i}(\eta) = 0 \qquad (2.10)$$

$$\sum_{i=1}^{n} \left.\frac{\partial\bar{x}^i}{\partial a_2}\right|_{\substack{a_1 = a_1^o \\ a_2 = a_2^o}} \frac{\partial}{\partial x^i}(\eta) = 0 \qquad (2.11)$$

A group with m transformations and r parameters has only the limited number of functionally independent invariants expressed by the following theorem (see Gaggioli and Moran [23]):

Theorem 2.4. Let a group of transformations be expressed by $\bar{x}_i = f_i(x_1,\ldots, x_m, a_1,\ldots, a_r)$ and $i = 1,\ldots,m$. Also define

$$\xi_j^{\ i} = \frac{\partial f_i}{\partial a_j}\bigg|_{\substack{a_1=a_1^{\circ}\\ \vdots \\ a_r=a_r^{\circ}}} . \tag{2.12}$$

The number of functionally independent invariants is equal to $m - \rho$ where ρ is the rank of the matrix whose elements are given by (2.12).

As an example consider the group given by (1.6). According to the theory presented above, the invariant η must satisfy the equation

$$- y \frac{\partial \eta}{\partial x} + x \frac{\partial \eta}{\partial y} = 0 \tag{2.13}$$

A general solution to (2.13) is $\eta = h_1(x^2 + y^2)$ where h_1 is an arbitrary function.

Similarity

When a differential equation is conformally invariant under a group, then the solutions of the transformed differential equation will have precisely the same form as that of the original equation. For example, if $u = u(x, y)$ is a solution to Laplace's equation, then $\bar{u} = \bar{u}(\bar{x}, \bar{y})$ is a solution of that equation under group (1.6) and $\bar{u}(\bar{x}, \bar{y})$ is the same function of $\bar{x}$ and $\bar{y}$ as $u(x, y)$ is of x and y. A solution which has this property is called an invariant solution with the understanding that a particular group of transformations is associated with it.

The group (2.2) has $n + m - 1$ functionally independent invariants which may be expressed by

$$\eta_1(x_1, \ldots, x_m), \ldots, \eta_{m-1}(x_1, \ldots, x_m) \qquad (2.14)$$

$$g_1(y_1, \ldots, y_n; x_1, \ldots, x_m), \ldots, g_n(y_1, \ldots, y_n; x_1, \ldots x_m) \qquad (2.15)$$

where the x's may be thought of as independent variables and the y's may be thought of as dependent variables. According to Morgan [8] the invariants have the property that the Jacobian J

$$J = \frac{\partial(g_1, \ldots, g_n)}{\partial(y_1, \ldots, y_n)} \qquad (2.16)$$

is not zero and the rank of the Jacobian matrix

$$\frac{\partial(\eta_1, \ldots, \eta_{m-1})}{\partial(x_1, \ldots, x_m)} \qquad (2.17)$$

is equal to m - 1.

For Laplace's equation under the group (1.6) the invariants are

$$\eta = h_1(x^2 + y^2)$$

$$g = u.$$

Suppose that an invariant solution of Laplace's equation is known and is denoted by $u = I(x, y)$. By definition of invariant solutions $\bar{u} = I(\bar{x}, \bar{y})$. Substitution of the invariant solution into the invariant $g = u$ implies that $\bar{u} = I(\bar{x}, \bar{y}) = u = I(x, y)$ and that I is an absolute invariant. The necessary and sufficient condition for I to be an invariant is

$$\left.\frac{\partial\bar{x}}{\partial a}\right|_{a=a^o} \frac{\partial I}{\partial x} + \left.\frac{\partial\bar{y}}{\partial a}\right|_{a=a^o} \frac{\partial I}{\partial y} = 0 \qquad (2.18)$$

where a^o is the identity element. For the group (1.6), (2.18) becomes

$$- y \frac{\partial I}{\partial x} + x \frac{\partial I}{\partial y} = 0. \tag{2.19}$$

The solution to (2.19) is

$$I = h_2(x^2 + y^2) = h_2(\eta) \tag{2.20}$$

where h_2 is an arbitrary function. Hence, the similarity variables for the problem are

$$u = h_2(\eta) \tag{2.21}$$

$$\eta = h_1(x^2 + y^2). \tag{2.22}$$

Letting $h_1(x^2 + y^2) = x^2 + y^2$ and transforming Laplace's equation with the similarity variables, we find

$$\eta \frac{\partial^2 h_1}{\partial \eta^2} + \frac{\partial h_1}{\partial \eta} = 0 \tag{2.23}$$

Equation (2.23) can be integrated by elementary methods and has the solution

$$h_1 = b_1 \ln(x^2 + y^2) + b_2 \tag{2.24}$$

where b_1 and b_2 are constants.

Before stating the general theory of similarity variables, let's summarize the procedure illustrated for Laplace's equation in more general form. First, referring to group (2.2) take an invariant $g_i = g_i(y_1, \ldots, y_n; x_1, \ldots, x_m)$ and substitute in it an invariant solution $y_i = I_i(x_1, \ldots, x_m)$ and observe that g_i becomes an invariant of the subgroup $\bar{x}_i$, $i = 1, \ldots, m$. Second, apply the necessary and sufficient condition expressed by theorem 2.2 and obtain the result that g_i becomes a function of $\eta_1, \ldots, \eta_m$. The similarity variables assume the form

$$\eta_i = \eta_i(x_1, \ldots, x_m) \tag{2.25}$$

$$g_i = g_i(\eta_1, \ldots, \eta_{m-1}) \tag{2.26}$$

where the functions g_i may be determined by substitution into the differential equation under consideration.

The above procedure is a general one and is placed on firm mathematical grounds by the following two theorems due to Morgan [8].

<u>Theorem 2.5</u>. Suppose we consider y_δ and $\bar{y}_\delta$ to be implicitly defined as functions of the x_i and $\bar{x}_i$ by the equations

$$z_\delta(x_1, \ldots, x_m) = g_\delta(y_1, \ldots, y_n; x_1, \ldots, x_m)$$

$$\bar{z}_\delta(\bar{x}_1, \ldots, \bar{x}_m) = g_\delta(\bar{y}_1, \ldots, \bar{y}_n; \bar{x}_1, \ldots, \bar{x}_m)$$

where g_δ are absolute invariants of a group. A necessary and sufficient condition for y_δ to be exactly the same functions of $x_1, \ldots, x_m$ as the $\bar{y}_\delta$ are of the $\bar{x}_1, \ldots, \bar{x}_m$ is that

$$z_\delta(x_1, \ldots, x_m) = \bar{z}_\delta(\bar{x}_1, \ldots, \bar{x}_m)$$

and

$$z_\delta(x_1, \ldots, x_m) = F_\delta(\eta_1, \ldots, \eta_{m-1})$$

where $\eta_1, \ldots, \eta_{m-1}$ are the invariants of the subgroup: $\bar{x}_i = \bar{x}_i(x_1, \ldots, x_m)$, $i = 1, \ldots, m$.

<u>Theorem 2.6</u>. If each of the differential forms ϕ_i (see (2.1)) in a system of partial differential equations is conformally invariant under the k^{th} enlargement of a group, then the invariant solution of that system can be expressed in terms of the solutions of a system of

the form

$$A_\delta(\eta_1, \ldots, \eta_{m-1}; F_1, \ldots, F_n; \frac{\partial^k F_1}{\partial \eta_1{}^k}, \ldots, \frac{\partial^k F_n}{\partial \eta_{m-1}{}^k}) = 0.$$

Boundary Conditions

As pointed out by Gaggioli and Moran, the auxiliary conditions must be compatible with invariant solutions. For an illustration of the concept consider Laplace's equation and denote the invariant solutions by $u = I(x, y)$ and $\bar{u} = I(\bar{x}, \bar{y})$. Suppose that the auxiliary conditions are

$$\Lambda(u, x, y) = 0 \text{ when } \Gamma(x, y) = 0. \qquad (2.27)$$

Transforming (2.27) under the group (1.6), we have

$$\Lambda(\bar{u}, \bar{x}, \bar{y}) = 0 \text{ when } \Gamma(\bar{x}, \bar{y}) = 0. \qquad (2.28)$$

Because in the group (1.6) $\bar{u} = u$, the above expression can be written as

$$\Lambda(I(x, y), x \cos a - y \sin a, x \sin a + y \cos a) = 0 \qquad (2.29)$$

when

$$\Gamma(x \cos a - y \sin a, x \sin a + y \cos a) = 0. \qquad (2.30)$$

The auxiliary conditions are said to be compatible if (2.27) agrees with (2.29) and (2.30) for all values of the parameter a.

As is easily seen by comparing (2.27) with (2.28), the boundary conditions are compatible under a group of transformations if they are invariant curves. Consequently, we may require that $\Lambda(u, x, y) = 0$ and $\Gamma(x, y) = 0$ are invariant curves and according to theorem 2.1 we can write (referring to group (1.6))

$$\left.\frac{\partial \bar{x}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial \Gamma}{\partial x} + \left.\frac{\partial \bar{y}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial \Gamma}{\partial y} = w_{\Gamma}(x, y)\ \Gamma(x, y) \qquad (2.31)$$

and

$$\left.\frac{\partial \bar{x}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial \Lambda}{\partial x} + \left.\frac{\partial \bar{y}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial \Lambda}{\partial y} + \left.\frac{\partial \bar{u}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial \Lambda}{\partial u} = w_{\Lambda}\ (x, y, u)\Lambda \qquad (2.32)$$

where w_{Γ} and w_{Λ} are undetermined functions of their respective arguments.

CHAPTER III

SIMILARITY BY INFINITESIMAL GROUPS

Infinitesimal Transformations

A set of infinitesimal transformations can be generated from a set of continuous transformations. For example, consider the continuous transformation

$$\begin{aligned} \bar{x} &= \xi(x, y, a) \\ \bar{y} &= \psi(x, y, a) \end{aligned} \qquad (3.1)$$

and let a^o be the identity element. Replacing the parameter a with a^o plus an infinitesimal δa and expanding in a Taylor series gives

$$\begin{aligned} \bar{x} &= \xi(x, y, a^o + \delta a) = x + \frac{\partial \xi}{\partial a}\bigg|_{a^o} \delta a + O(\delta a^2) \\ \bar{y} &= \psi(x, y, a^o + \delta a) = y + \frac{\partial \psi}{\partial a}\bigg|_{a^o} \delta a + O(\delta a^2). \end{aligned} \qquad (3.2)$$

The system (3.2) can be thought of as an infinitesimal transformation. By again using Taylor series, the effect of an infinitesimal transformation on a function $\omega(x, y)$ is seen to be

$$\omega(\bar{x}, \bar{y}) = \omega(x, y) + [\frac{\partial \xi}{\partial a}\bigg|_{a^o} \frac{\partial \omega}{\partial x} + \frac{\partial \psi}{\partial a}\bigg|_{a^o} \frac{\partial \omega}{\partial y}]\delta a + \ldots \quad .$$

Let $X = \frac{\partial \xi}{\partial a}\Big|_{a^o}$ and $Y = \frac{\partial \psi}{\partial a}\Big|_{a^o}$; then the second term in the above series can be written as $Q\omega\delta a$ where $Q\omega \equiv X \frac{\partial \omega}{\partial x} + Y \frac{\partial \omega}{\partial y}$. Once the infinitesimal transformation is known the term $Q\omega$ can be written and for this reason $Q\omega$ is said to represent it. Sometimes in the literature $Q\omega$ is called an <u>infinitesimal transformation</u> with the understanding that it represents one only in the above sense.

Using the notation introduced above and letting $\delta a = \varepsilon$ the system (3.2) becomes

$$\bar{x} = x + X\varepsilon + 0(\varepsilon^2)$$
$$\bar{y} = y + Y\varepsilon + 0(\varepsilon^2). \tag{3.3}$$

Let us investigate whether the transformation (3.3) is a group to order ε^2. The identity element is $\varepsilon = 0$ and the Jacobian of the transformation is

$$\begin{vmatrix} \frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{x}}{\partial y} \\ \frac{\partial \bar{y}}{\partial x} & \frac{\partial \bar{y}}{\partial y} \end{vmatrix} = \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial y} - \frac{\partial \bar{y}}{\partial x}\frac{\partial \bar{x}}{\partial y}$$
$$= 1 + \varepsilon(X_x + Y_y) + 0(\varepsilon^2) \neq 0.$$

Hence the inverse transformation always exists. Letting $\bar{x}_1 = x + X\varepsilon_1$ and $\bar{y}_1 = y + Y\varepsilon_2$ and substituting into $\bar{x}_2 = x + X\varepsilon_2$, we obtain

$$\bar{x}_2(\bar{x}_1, \bar{y}_1) = x + X\varepsilon_1 + X(x + \varepsilon_1 X, y + \varepsilon_1 Y)\varepsilon_2$$
$$= x + X\varepsilon_1 + X\varepsilon_2 + 0(\varepsilon_1 \varepsilon_2).$$

Because ε_1 and ε_2 are arbitrarily small infinitesimals, their sum is an infinitesimal and successive transformations lead to a member of the group. All properties are satisfied and the system (3.3) is a group.

As has been noted above an infinitesimal transformation can be generated from a continuous function and in particular from a continuous group of transformations. For example, consider the group

$$\begin{aligned} \bar{x} &= e^{a}x + e^{a}(1 - e^{a})y \\ \bar{y} &= e^{2a}\, y. \end{aligned} \qquad (3.4)$$

The corresponding infinitesimal transformation is

$$\begin{aligned} \bar{x} &= x + (x - y)\varepsilon \\ \bar{y} &= y + 2y\, \varepsilon. \end{aligned} \qquad (3.5)$$

This question arises: Once the system (3.5) is known, can the system (3.4) be found? There are two procedures illustrated by Cohen [24] and one will be demonstrated here. To find the system (3.4) from (3.5) solve the following set of equations:

$$\frac{d\bar{x}}{\bar{x}-\bar{y}} = \frac{d\bar{y}}{2\bar{y}} = \frac{da}{1} \,. \qquad (3.6)$$

As can be easily checked by the method presented in Chapter II, the constant η of integration for the first pair of equations is an invariant of the systems (3.4) and (3.5) and is found to be

$$\eta = \frac{\bar{x}+\bar{y}}{\bar{y}^{\frac{1}{2}}} = \frac{x+y}{y^{\frac{1}{2}}} \qquad (3.7)$$

Integrating the second pair of equations (3.6), we have

$$\ln \bar{y} = 2a$$

or

$$\frac{\bar{y}}{e^{2a}} = c_2 \qquad (3.8)$$

where c_2 is an integration constant. Because at $a = 0$ $\bar{y} = y$, then

$c_2 = y$. Equations (3.7) and (3.8) are simultaneous equations representing the equivalent finite continuous transformation which is generated by the infinitesimal group and their solution is easily found to be the system (3.4).

Invariants

The necessary and sufficient condition for a function $\eta(x, y)$ to be invariant under an infinitesimal transformation is the same as for continuous transformations and is given by

$$Q\eta = 0. \tag{3.9}$$

Also, as in the case of continuous transformations, the necessary and sufficient condition for a curve $H(x, y) = 0$ to be an invariant curve under an infinitesimal transformation is

$$QH = w(x, y)H \tag{3.10}$$

where $w(x, y)$ is an unspecified function. Considering x and y as independent variables and u as a dependent variable, an infinitesimal transformation can be written as

$$\begin{aligned} \bar{x} &= x + X\epsilon + O(\epsilon^2) \\ \bar{y} &= y + Y\epsilon + O(\epsilon^2) \\ \bar{u} &= u + U\epsilon + O(\epsilon^2). \end{aligned} \tag{3.11}$$

If we append to (3.11) the partial derivatives of $\bar{u}$ with respect to $\bar{x}$ and $\bar{y}$ up to and including the k^{th} derivative, then we obtain another infinitesimal group which is called the k^{th} enlargement. Using a sufficient enlargement of the group (3.11) the conditions expressed by (3.9) and (3.10) can be written for a differential equation of the form

$$\phi(x,\ y,\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ldots,\ \frac{\partial^k u}{\partial x^i \partial y^j}) = 0 \qquad (3.12)$$

where $i + j = k$. Letting z_1 through z_n denote the arguments of (3.12) and Z_1 through Z_n be the corresponding coefficients of ε in the k^{th} enlargement of the group (3.11), then (3.10) can be written as

$$Z_1 \frac{\partial \phi}{\partial z_1} + Z_2 \frac{\partial \phi}{\partial z_2} + \ldots + Z_n \frac{\partial \phi}{\partial z_n} = w_1(z_1,\ldots,\ z_n)\phi. \qquad (3.13)$$

Equation (3.13) is then a necessary and sufficient condition for the ϕ to be an invariant curve.

Canonical Variables

As was mentioned earlier an infinitesimal transformation may be represented as $Q\omega$ and referring to the system (3.3)

$$Q = X \frac{\partial}{\partial x}\ (\ ) + Y \frac{\partial}{\partial y}\ (\ )\ . \qquad (3.14)$$

With (3.14) we see that

$$\begin{aligned} Qx &= X \\ Qy &= Y\ . \end{aligned} \qquad (3.15)$$

Consider a change of variable which is denoted by

$$\begin{aligned} x_1 &= x_1(x,\ y) \\ y_1 &= y_1(x,\ y). \end{aligned} \qquad (3.16)$$

Then

$$Q\,x_1 = X\,\frac{\partial x_1}{\partial x} + Y\frac{\partial x_1}{\partial y} = X_1$$
$$Q\,y_1 = X\,\frac{\partial y_1}{\partial x} + Y\frac{\partial y_1}{\partial y} = Y_1 \tag{3.17}$$

and the new infinitesimal transformation under the change of variables becomes

$$Q\,\omega_1 = X_1\,\frac{\partial \omega_1}{\partial x_1} + Y_1\,\frac{\partial \omega_1}{\partial y_1} \tag{3.18}$$

where ω_1 is a function of x_1 and y_1. This procedure allows us to make a change of variables and find the representation of the new infinitesimal transformation by solving the system of equations (3.17).

As an illustration consider the rotation group (1.6) and the change of variables defined by

$$\rho = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x). \tag{3.19}$$

Since for the rotation group $Q \equiv -y\,\frac{\partial}{\partial x}(\ ) + x\,\frac{\partial}{\partial y}(\ )$, we find that

$$Q\rho = -y\,\frac{\partial \rho}{\partial x} + x\,\frac{\partial \rho}{\partial y}$$
$$= -y\,x\,(x^2 + y^2)^{-\frac{1}{2}} + y\,x\,(x^2 + y^2)^{+\frac{1}{2}} = 0$$

and

$$Q\theta = -y\,\frac{\partial \theta}{\partial x} + x\,\frac{\partial \theta}{\partial y}$$
$$= -\frac{y^2}{x^2-y^2} + \frac{x^2}{x^2-y^2} = 1.$$

Hence the new infinitesimal transformation is represented by

$$Q\omega(\rho,\,\theta) = \frac{\partial \omega}{\partial \theta}$$

and the new infinitesimal transformation is

$$\bar{\theta} = \theta + \varepsilon$$
$$\bar{\rho} = \rho \ . \tag{3.20}$$

It is interesting to note that the invariant of the group (3.20) is ρ.

The variables which reduce an infinitesimal transformation to the form $Q\omega_1 = \frac{\partial \omega_1}{\partial y_1}$ or $Q\omega_1 = \frac{\partial \omega_1}{\partial x_1}$ are called canonical variables and the resulting infinitesimal group is said to be in canonical form. Following this notation the transformation (3.19) is a set of canonical variables for the rotation group.

Reduction of the Number of Independent Variables in a Partial Differential Equation.

Letting z_1 through z_n denote the arguments of the differential equation (3.12) and rewriting, we have

$$\phi(z_1, z_2, \ldots, z_n) = 0. \tag{3.21}$$

The infinitesimal transformations for the z's may be written symbolically as

$$\bar{z}_i = z_i + Z_i \varepsilon \quad i = 1, \ldots, n \tag{3.22}$$

The infinitesimal representation of (3.22) is

$$Q\phi = Z_1 \frac{\partial \phi}{\partial z_1} + \ldots + Z_n \frac{\partial \phi}{\partial z_n} \ . \tag{3.23}$$

As shown in the previous section of this chapter we may find a set of canonical variables which will reduce the transformation (3.22) to canonical form. Let v_i $i = 1, \ldots, n$ denote the appropriate canonical variables and let the canonical form of the group (3.22) be denoted by

$$\bar{v}_1 = v_1 + \varepsilon$$
$$\bar{v}_i = v_i \quad i = 2, \ldots, n. \tag{3.24}$$

If the differential equation (3.21) is conformally invariant (see Chapter II) to the order ε^2 when transformed by the group (3.22), then according to theorem (2.1) of chapter II we have

$$Z_1 \frac{\partial\phi}{\partial z_1} + \ldots + Z_n \frac{\partial\phi}{\partial z_n} = w(z_1, \ldots, z_n)\, \phi(z_1, \ldots, z_n). \tag{3.25}$$

Introducing the canonical variables v_1 we find that (3.25) becomes

$$\frac{\partial\phi_1}{\partial v_1} = w_2(v_1, \ldots, v_n)\, \phi_1\, (v_1, \ldots, v_n) \tag{3.26}$$

where ϕ_1 is the differential equation ϕ when transformed by the canonical variables. Also since v_i, $i = 2, \ldots, n$ are absolute invariants of the group (3.24), we may apply the necessary and sufficient condiditions given by theorem (2.2) to find that

$$\frac{\partial v_i}{\partial v_1} = 0 \quad i = 2, \ldots, n. \tag{3.27}$$

If ϕ_1 is absolutely invariant under the group (3.24), then (3.26) becomes

$$\frac{\partial\phi_1}{\partial v_1} = 0. \tag{3.28}$$

Equations (3.27) and (3.28) guarantee that the variable v_1 does not appear either implicitly or explicitly in ϕ_1. Consequently, the number of variables has been reduced by one. If z_1 were an independent variable the number of such variables would be reduced by one.

This result is precisely the one needed to obtain similarity solutions.

Example

The infinitesimal transformations corresponding to the rotation group (1.6) are

$$\bar{x} = x - y\varepsilon$$
$$\bar{y} = y + x\varepsilon \qquad (3.29)$$
$$\bar{u} = u$$

Laplace's equation in two dimensions is absolutely invariant under the group (3.29). We seek canonical variables s_1, s_2, and q_1 such that

$$\bar{s}_1(\bar{x}, \bar{y}) = s_1(x, y) + \varepsilon$$
$$\bar{s}_2(\bar{x}, \bar{y}) = s_2(x, y) \qquad (3.30)$$
$$q_1(\bar{x}, \bar{y}) = q_1(x, y).$$

An appropriate choice for s_1 and s_2 is given by letting $s_1 = \theta$ and $s_2 = \rho$ in (3.19). Inspection of (3.29) shows that u is an appropriate choice for q_1. With these choices (3.30) becomes

$$\bar{\theta} = \theta + \varepsilon$$
$$\bar{\rho} = \rho \qquad (3.31)$$
$$\bar{u} = u$$

In group (3.31) u is an invariant and accordingly $\frac{\partial u}{\partial \theta} = 0$. Treating θ and ρ as new independent variables, Laplace's equation transforms to

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho}\right) = 0. \qquad (3.32)$$

The similarity variables generated by the group (3.29) are

$$\rho = \sqrt{x^2 + y^2} \qquad u = u(\rho) \tag{3.33}$$

As is seen from this example, the process of introducing polar coordinates and requiring that $\frac{\partial u}{\partial \theta} = 0$ is the same as searching for similarity variables under the group of rotations.

Similarity Solutions in Practice

In practice we may construct similarity solutions by a procedure described by Bluman and Cole [27]. The method will be described for the case of two independent variables and two dependent variables; however, the ideas are applicable to other cases.

The infinitesimal transformation is assumed to have the form

$$\begin{aligned} \bar{x} &= x + X(x, y)\varepsilon + O(\varepsilon^2) \\ \bar{y} &= y + Y(x, y)\varepsilon + O(\varepsilon^2) \\ \bar{u} &= u + U(u, v, x, y)\varepsilon + O(\varepsilon^2) \\ \bar{v} &= v + V(u, v, x, y)\varepsilon + O(\varepsilon^2) \end{aligned} \tag{3.34}$$

and the independent variables are x and y while the dependent variables are u and v. The differential equation may be represented by

$$\phi(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^k u}{\partial x^i y^j}, \frac{\partial^k v}{\partial x^n \partial y^l}) = 0 \tag{3.35}$$

where $i + j = k$ and $n + l = k$. We may append to the group (3.34) the transformations of the derivatives of u and v which appear in ϕ. As the determination of the transform of a derivative requires some careful manipulations, the transform for $\frac{\partial u}{\partial x}$ will be presented here. By the chain rule

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial \bar{u}}{\partial x} \frac{\partial x}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial y} \frac{\partial y}{\partial \bar{x}} . \tag{3.36}$$

To apply (3.36) we must determine $\frac{\partial x}{\partial \bar{x}}$ and $\frac{\partial y}{\partial \bar{x}}$ remembering that terms of order ε^2 and higher are dropped. Differentiating the first of equations (3.34) with respect to $\bar{x}$, we have

$$1 = \frac{\partial x}{\partial \bar{x}} + \frac{\partial}{\partial \bar{x}} (X)\varepsilon + O(\varepsilon^2). \tag{3.37}$$

From (3.36) we see that

$$\frac{\partial}{\partial \bar{x}} (\ ) = \frac{\partial}{\partial x} (\ ) \frac{\partial x}{\partial \bar{x}} + \frac{\partial}{\partial y} (\ ) \frac{\partial y}{\partial \bar{x}} . \tag{3.38}$$

Differentiating the second of (3.34) with respect to $\bar{x}$ and using (3.38), we have

$$0 = \frac{\partial y}{\partial \bar{x}} + \frac{\partial}{\partial \bar{x}} (Y) \varepsilon + O(\varepsilon^2)$$

or

$$\frac{\partial y}{\partial \bar{x}} = - (Y_x \frac{\partial x}{\partial \bar{x}} + Y_y \frac{\partial y}{\partial \bar{x}}) \varepsilon + O(\varepsilon^2). \tag{3.39}$$

The above equation tells us that $\frac{\partial y}{\partial \bar{x}}$ is at most of order ε and $\frac{\partial y}{\partial \bar{x}} \varepsilon$ is at most of order (ε^2). Hence (3.37) becomes

$$1 = \frac{\partial x}{\partial \bar{x}} + \varepsilon X_x \frac{\partial x}{\partial \bar{x}} + O(\varepsilon^2)$$

or

$$\frac{\partial x}{\partial \bar{x}} = 1 - \varepsilon \frac{\partial x}{\partial \bar{x}} X_x + O(\varepsilon^2). \tag{3.40}$$

The above expression tells us that $\frac{\partial x}{\partial \bar{x}} = 1 + O(\varepsilon)$ and that we may further simplify to obtain

$$\frac{\partial x}{\partial \bar{x}} = 1 - \varepsilon(X_x) + O(\varepsilon^2). \tag{3.41}$$

Similarly

$$\frac{\partial y}{\partial \bar{x}} = - \varepsilon\ Y_x + O(\varepsilon^2). \tag{3.42}$$

Substituting (3.41) and (3.42) into (3.36) gives

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial \bar{u}}{\partial x}(1 - \varepsilon\ X_x) + \frac{\partial \bar{u}}{\partial y}(- \varepsilon\ Y_x) + O(\varepsilon^2). \tag{3.43}$$

From (3.34)

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial u}{\partial x} + U_x \varepsilon + O(\varepsilon^2) \tag{3.44}$$

and

$$\frac{\partial \bar{u}}{\partial y} = \frac{\partial u}{\partial y} + \varepsilon\ U_y + O(\varepsilon^2). \tag{3.45}$$

Substituting (3.44) and (3.45) into (3.43) gives

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial u}{\partial x} + \varepsilon[- X_x\ u_x + U_x - Y_x\ u_y] + O(\varepsilon^2). \tag{3.46}$$

Having calculated all derivatives in the barred variables and letting $\bar{\phi}$ denote the differential equation (3.35) in the barred vari ables, we require that X, U, Y, and V be chosen such that

$$\bar{\phi} = (1 + \varepsilon\ \bar{\Omega}(x, y, u, v))\phi. \tag{3.47}$$

Equation (3.47) is the necessary condition that ϕ be conformally invariant under the group (3.34). If $u = \theta_1(x, y)$ and $v = \theta_2(x, y)$ is a solution to ϕ, then $\theta_1(\bar{x}, \bar{y})$ and $\theta_2(\bar{x}, \bar{y})$ is a solution to $\bar{\phi}$ by virtue of (3.47). Using (3.34) we see that

$$\bar{u} = \theta_1(\bar{x}, \bar{y}) = \theta_1(x, y) + U\varepsilon \tag{3.48}$$

Expanding (3.48) in a Taylor series we have

$$\theta_1(x, y) + (X\, u_x + Y\, u_y)\, \varepsilon = \theta_1(x, y) + U\varepsilon + O(\varepsilon^2)$$

and we conclude that

$$X\, u_x + Y\, u_y = U. \tag{3.49}$$

Similarly we find that

$$X\, v_x + Y\, v_y = V \tag{3.50}$$

Once U, V, X, and Y have been determined to satisfy (3.47), then (3.49) and (3.50) define the similarity variables for the differential equation ϕ. Actually (3.49) and (3.50) can be used when satisfying the invariance condition expressed by (3.47). According to the terminology of Bluman and Cole [27] the similarity variables obtained are called classical if (3.49) and (3.50) are not used in (3.47) and non-classical if they are used. All similarity variables generated in this thesis by infinitesimal methods will be the non-classical type.

CHAPTER IV

APPLICATION OF THE FINITE GROUP METHOD

Burgers' Equation

The Burgers' equation $u_y + uu_x = u_{xx}$ has been used as a mathematical model of turbulence [30] and in the approximate theory for weak nonstationary shock waves in a real fluid [31]. We seek groups of transformations of the form

$$\begin{aligned} \bar{x} &= f_1(x, y, a) \\ \bar{y} &= f_2(x, y, a) \\ \bar{u} &= f_4(a)\, u + f_5(x, y, u, a) \end{aligned} \tag{4.1}$$

such that the Burgers' equation is conformally invariant. To accomplish this task, we append to (4.1) the relevant partial derivatives which are

$$\begin{aligned} \frac{\partial \bar{u}}{\partial \bar{x}} &= \left(f_4 \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial x} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial x}\right) \frac{\partial x}{\partial \bar{x}} \\ &\quad + \left(f_4 \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial y} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial y}\right) \frac{\partial y}{\partial \bar{x}} \end{aligned} \tag{4.2}$$

$$\begin{aligned} \frac{\partial \bar{u}}{\partial \bar{y}} &= \left(f_4 \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial x} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial x}\right) \frac{\partial x}{\partial \bar{y}} \\ &\quad + \left(f_4 \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial y} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial y}\right) \frac{\partial y}{\partial \bar{y}} \end{aligned} \tag{4.3}$$

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \frac{\partial^2 x}{\partial \bar{x}^2} (f_4 \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial x} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial x})$$

$$+ \frac{\partial^2 y}{\partial \bar{x}^2} (f_4 \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial y} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial y})$$

$$+ (\frac{\partial x}{\partial \bar{x}})^2 (f_4 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f_5}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \frac{\partial f_5}{\partial u} + 2 \frac{\partial^2 f_5}{\partial u \partial x} \frac{\partial u}{\partial x})$$

$$+ \frac{\partial x}{\partial \bar{x}} \frac{\partial y}{\partial \bar{x}} (2 f_4 \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 f_5}{\partial x \partial y} + 2 \frac{\partial f_5}{\partial u} \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 f_5}{\partial u \partial y} \frac{\partial u}{\partial x}$$

$$+ 2 \frac{\partial^2 f_5}{\partial u \partial x} \frac{\partial u}{\partial y})$$

$$+ (\frac{\partial y}{\partial \bar{x}})^2 [f_4 \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 f_5}{\partial y^2} + \frac{\partial f_5}{\partial u} \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial^2 f_5}{\partial u \partial y} \frac{\partial u}{\partial y}] \quad . \quad (4.4)$$

In (4.4) it must be assumed that $\frac{\partial^2 f_5}{\partial u^2} = 0$; otherwise the square of the derivatives of u would appear on the right-hand side of (4.4) and would force this assumption in the present example. In problems where the squares of derivatives appear in the differential equation, such terms should be retained. Using (4.1),(4.2), (4.3) and (4.4) we form the Burgers' equation in the barred variables and equate it to zero. Since we cannot place any restrictions on u or its derivatives, the coefficients of such terms must vanish if they do not appear in the differential equation. Inspection of (4.4) reveals that the coefficient of $\frac{\partial^2 u}{\partial x \partial y}$ is equal to

$$- 2 f_4 \frac{\partial x}{\partial \bar{x}} \frac{\partial y}{\partial \bar{x}} - 2 \frac{\partial f_5}{\partial u} \frac{\partial x}{\partial \bar{x}} \frac{\partial y}{\partial \bar{x}}$$

and for this to vanish, either $\frac{\partial y}{\partial \bar{x}} = 0$ or $\frac{\partial f_5}{\partial u} = -f_4$. The latter alternative is rejected since it would cause $\bar{u}$ to vanish identically.

The coefficient of u, which arises from the term $\bar{u}\ \bar{u}_{\bar{x}}$, is

$$f_4 \frac{\partial f_5}{\partial x} \frac{\partial x}{\partial \bar{x}}$$

which forces $\frac{\partial f_5}{\partial x} = 0$. With the above observation, a search for terms which do not have u or its derivatives as coefficients reveals that

$$\frac{\partial f_5}{\partial y} \frac{\partial y}{\partial \bar{y}} = 0$$

which forces f_5 to be a constant.

The coefficient of $\frac{\partial u}{\partial x}$ is

$$f_4 \frac{\partial x}{\partial \bar{y}} + f_4\ f_5 \frac{\partial x}{\partial \bar{x}} - f_4 \frac{\partial^2 x}{\partial \bar{x}^2} = 0. \tag{4.5}$$

The final requirement is that the coefficients of the terms which appear in the differential equation be equated; so that

$$+ f_4{}^2 \frac{\partial x}{\partial \bar{x}} = f_4 \frac{\partial y}{\partial \bar{y}} = + f_4 \left(\frac{\partial x}{\partial \bar{x}}\right)^2. \tag{4.6}$$

From (4.6) we have

$$\frac{\partial x}{\partial \bar{x}} = f_4$$
$$\frac{\partial y}{\partial \bar{y}} = f_4 \frac{\partial x}{\partial \bar{x}} = f_4{}^2 \tag{4.7}$$

Substituting the results of (4.7) into (4.5) gives

$$\frac{\partial x}{\partial \bar{y}} = - f_5\ f_4 \tag{4.8}$$

Integrating (4.7) and (4.8) yields

$$x = f_4\,\bar{x} - f_5\,f_4\,\bar{y}^{\,*} \qquad y = f_4^{\,2}\,\bar{y} \tag{4.9}$$

The inverse transformation of (4.9) is

$$\bar{x} = \frac{1}{f_4}\,x + \frac{f_5}{f_4^{\,2}}\,y \qquad \bar{y} = \frac{1}{f_4^{\,2}}\,y \tag{4.10}$$

The transformed Burgers' equation is

$$\bar{u}_{\bar{y}} + \bar{u}\bar{u}_{\bar{x}} - \bar{u}_{\bar{x}\bar{x}} = f_4^{\,3}\,(u_y + uu_x - u_{xx})$$

which satisfies the conformal invariance requirement. For convenience let $f_4 = \frac{1}{f}$ and $H(a) = f_5(a)\,f^2(a)$. Using (4.10) and (4.1) the transformation group becomes

$$\bar{x} = f\,x + Hy \qquad \bar{y} = f^2 y \qquad \bar{u} = \frac{1}{f}\,u + \frac{H}{f^2}\,. \tag{4.11}$$

The transformation (4.11) must satisfy the closure property defined in Chepter II. Individual transformations can be denoted by

$$\bar{x}_i \equiv f(a_i)x + H(a_i)y \qquad \bar{y}_i \equiv f^2(a_i)y \qquad \bar{u}_i \equiv \frac{1}{f(a_i)}\,u + \frac{H(a_i)}{f^2(a_i)} \tag{4.12}$$

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* A constant multiplied by a function of the parameter <u>a</u> could be added to obtain a more general group.

where a_i is any real number. The closure property requires that if $\bar{x}_1$ and $\bar{x}_2$ are transformations then there exists an $\bar{x}_3$ such that

$$\bar{x}_2(\bar{x}_1, \bar{y}_1, a_2) = \bar{x}_3(x, y, a_3). \tag{4.13}$$

Similarly,

$$\bar{y}_2(\bar{x}_1, \bar{y}_1, a_2) = \bar{y}_3(x, y, a_3) \tag{4.14}$$

$$\bar{u}_2(\bar{u}_1, a_2) = \bar{u}_3(u, a_3). \tag{4.15}$$

Substituting (4.12) into (4.13), (4.14) and (4.15)

$$\begin{aligned}\bar{x}_3 &= f(a_2)\,[f(a_1)\,x + H(a_1)\,y] + H(a_2)\,f^2(a_1)y \\ &= f(a_3)\,x + H(a_3)y\end{aligned} \tag{4.16}$$

$$\bar{y}_3 = f^2(a_2)\,(f^2(a_1)\,x) = f^2(a_3)y \tag{4.17}$$

$$\begin{aligned}\bar{u}_3 &= \frac{1}{f(a_2)}\left(\frac{u}{f(a_1)} + \frac{H(a_1)}{f^2(a_1)}\right) + \frac{H(a_2)}{f^2(a_2)} \\ &= \frac{u}{f(a_3)} + \frac{H(a_3)}{f^2(a_3)}\,.\end{aligned} \tag{4.18}$$

The above three equations require that

$$H(a_3) = f(a_2)\,H(a_1) + H(a_2)\,f^2(a_1) \tag{4.19a}$$

$$f(a_3) = f(a_1)\,f(a_2)\,. \tag{4.19b}$$

Equation (4.19b) implies that interchanging a_1 and a_2 does not change a_3 because the real numbers are commutative. Using this result in (4.19a) gives

$$H(a_3) = f(a_2)\,H(a_1) + H(a_2)\,f^2(a_1) = f(a_1)\,H(a_2) + H(a_1)\,f^2(a_2)$$

which may be rearranged as

$$H(a_1)\ f(a_2)\ [1 - f^2(a_2)] = H(a_2)[f(a_1)\ (1 - f^2(a_1)]. \tag{4.20}$$

Equation (4.20) is satisfied if $H(a) = f(a)\ (1 - f(a))$ and (4.11) becomes

$$\begin{aligned}\bar{x} &= f(a)\ x + f(a)\ (1 - f(a))y \\ \bar{y} &= f^2(a)y \\ \bar{u} &= \frac{u}{f(a)} + (1 - f(a))\ /\ f(a).\end{aligned} \tag{4.21}$$

Satisfying the closure property has produced a result which immediately satisfies the existence of an identity element a_o if $f(a_o) = 1$. Since equation (4.21) is seen to possess an inverse, it satisfies all the requirements of a group.

The invariants of (4.12) are described by functionally independent solutions of

$$\frac{\partial\bar{x}}{\partial a}\bigg|_{a=a_o} \frac{\partial\lambda_i}{\partial x} + \frac{\partial\bar{y}}{\partial a}\bigg|_{a=a_o} \frac{\partial\lambda_i}{\partial y} + \frac{\partial\bar{u}}{\partial a}\bigg|_{a=a_o} \frac{\partial\lambda_i}{\partial u} = 0. \tag{4.22}$$

Substituting (4.21) into (4.22) gives

$$f'(a_o)\ (x-y)\ \frac{\partial\lambda_i}{\partial x} + 2\ f'(a_o)y\ \frac{\partial\lambda_i}{\partial y} - f'(a_o)\ (u+1)\ \frac{\partial\lambda_i}{\partial u} = 0. \tag{4.23}$$

Assuming that $f'(a_o)$ does not vanish and that $\lambda_1 = \lambda_1(x, y)$*, we have

$$(x - y)\ \frac{\partial\lambda_1}{\partial x} + 2y\ \frac{\partial\lambda_1}{\partial y} = 0. \tag{4.24}$$

The solution of (4.24) is found by the method of Lagrange [32] (or by separation of variables) which forms the subsystem of equations

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* The subgroup $\bar{x}$ and $\bar{y}$ possesses an invariant.

$$\frac{dx}{x-y} = \frac{dy}{2y} = \frac{d\lambda_1}{0}\,. \qquad (4.25)$$

The solution to (4.25) is

$$\lambda_1 = \lambda_1[\frac{x+y}{y^{\frac{1}{2}}}]. \qquad (4.26)$$

A second invariant of (4.21) is found by letting $\lambda_2 = \lambda_2(x, y, u)$ and (4.23) becomes

$$(x - y)\,\frac{\partial\lambda_2}{\partial x} + 2y\,\frac{\partial\lambda_2}{\partial y} - (u + 1)\,\frac{\partial\lambda_2}{\partial u} = 0. \qquad (4.27)$$

Again using the method of Lagrange the solution of (4.27) is found to be

$$\lambda_2 = \lambda_2(\frac{x+y}{y^{\frac{1}{2}}}\,,\ (u + 1)y^{\frac{1}{2}}) \qquad (4.28)$$

Equations (4.26) and (4.28) implicitly define the sought similarity variables. A special case is

$$\lambda_1 = \eta = \frac{x+y}{y^{\frac{1}{2}}} \qquad (4.29)$$

$$\lambda_2 = (u + 1)y^{\frac{1}{2}} = F(\eta). \qquad (4.30)$$

Substituting (4.29) and (4.30) into the Burgers' equation gives

$$F'' - FF' + \tfrac{1}{2}\,\eta F' + \tfrac{1}{2}\,F = 0 \qquad (4.31)$$

or

$$\frac{d}{d\eta}\,[F' - \tfrac{1}{2}\,F^2 + \tfrac{1}{2}\,\eta F] = 0.$$

One integration results in

$$F' - \tfrac{1}{2}\,F^2 + \tfrac{1}{2}\,\eta F = \text{const}. \qquad (4.32)$$

which is a form of the Riccati equation (see ref. [33]) and it may be transformed into a linear second order differential equation by the transformation

$$F = -2\,\frac{\psi_\eta}{\psi}\,. \tag{4.33}$$

Applying (4.33) to (4.32) results in

$$2\,\psi_{\eta\eta} + \eta\,\psi_\eta + C_1\,\psi = 0. \tag{4.34}$$

The solution to (4.34) is found by standard methods (see [34]) and is

$$\psi_1 = C_{10}\;{}_1F_1\left(\frac{c_1}{2};\tfrac{1}{2};z_1\right) + C_{11}\,x^{\frac{1}{2}}\;{}_1F_1\left(\tfrac{1}{2}+\frac{c_1}{2};\,{}^3/2;\,z_1\right) \tag{4.35}$$

where

$$\begin{aligned} \psi(\eta) &= \psi(z_1) \\ z_1 &= -\tfrac{1}{4}\,\eta^2 \end{aligned} \tag{4.36}$$

$$\begin{aligned} {}_1F_1(c_2, b, z_1) &= \sum_{r=0}^{\infty} \bar{A}_r (z_1)^r \\ \bar{A}_r &= \frac{c_2(c_2+1)\ \ldots\ (c_2+r-1)}{b(b+1)\ \ldots\ (b+r-1)r\,!} \end{aligned} \tag{4.37}$$

Laminar Boundary Layer

The boundary layer equations for steady, incompressible two-dimensional flow (see Schlichting [35]) are

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = U_\infty\,\frac{dU_\infty}{dx} + \nu\,\frac{\partial^2 u^*}{\partial y^{*2}} \tag{4.38a}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0. \tag{4.38b}$$

A set of dimensionless coordinates is defined by

$$y^* = \frac{L}{\sqrt{\frac{U_c L}{\nu}}}\, y \qquad v^* = \frac{U_c}{\sqrt{\frac{U_c L}{\nu}}}\, v \tag{4.39}$$

$$x^* = L\, x \qquad u^* = U_c\, u,$$

where U_c is a characteristic velocity, L is a characteristic length, and ν is the kinematic viscosity. Introducing equation (4.39) into (4.38) we have

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \frac{U_\infty}{U_c^{\,2}}\,\frac{dU_\infty}{dx} + \frac{\partial^2 u}{\partial y^2} \tag{4.40a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.40b}$$

We want to determine continuous one parameter groups of transformations which will keep (4.40a) and (4.40b) conformally invariant. For reasons which will become clear later, the term $\frac{U_\infty}{U_c^{\,2}}\,\frac{dU_\infty}{dx}$ will be assumed to be a function of x and y denoted by P(x, y). A general group

$$\bar{u} = f_4(a)u + f_5(u,\ v,\ x,\ y,\ a)$$

$$\bar{v} = f_6(a)v + f_7(u,\ v,\ x,\ y,\ a) \tag{4.41}$$

$$\bar{x} = f_1(x,\ y,\ a)$$

$$\bar{y} = f_2(x,\ y,\ a)$$

is assumed. The relevant derivatives are

$$\frac{\partial \bar{u}}{\partial \bar{x}} = (f_4 \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial x} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial v} \frac{\partial v}{\partial x}) \frac{\partial x}{\partial \bar{x}} + (f_4 \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial y} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial v} \frac{\partial v}{\partial y}) \frac{\partial y}{\partial \bar{x}} \tag{4.42}$$

$$\frac{\partial \bar{v}}{\partial \bar{y}} = (f_6 \frac{\partial v}{\partial x} + \frac{\partial f_7}{\partial x} + \frac{\partial f_7}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f_7}{\partial v} \frac{\partial v}{\partial x}) \frac{\partial x}{\partial \bar{y}} + (f_6 \frac{\partial v}{\partial y} + \frac{\partial f_7}{\partial y} + \frac{\partial f_7}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f_7}{\partial v} \frac{\partial v}{\partial y}) \frac{\partial y}{\partial \bar{y}} \tag{4.43}$$

$$\frac{\partial \bar{u}}{\partial \bar{y}} = (f_4 \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial x} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial v} \frac{\partial v}{\partial x}) \frac{\partial x}{\partial \bar{y}} \quad (f_4 \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial y} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial v} \frac{\partial v}{\partial y}) \frac{\partial y}{\partial \bar{y}} \tag{4.44}$$

$$\begin{aligned}
\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} &= (f_4 \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial x} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f_5}{\partial v} \frac{\partial v}{\partial x}) \frac{\partial^2 x}{\partial \bar{y}^2} \\
&+(f_4 \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial u} + \frac{\partial f_5}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial v} \frac{\partial v}{\partial y}) \frac{\partial^2 y}{\partial \bar{y}^2} \\
&+ (f_4 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f_5}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \frac{\partial f_5}{\partial u} + 2 \frac{\partial u}{\partial x} \frac{\partial^2 f_5}{\partial u \partial x} + \frac{\partial f_5}{\partial v} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 f_5}{\partial v \partial x} \frac{\partial v}{\partial x}) (\frac{\partial x}{\partial \bar{y}})^2 \\
&+(2f_4 \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 f_5}{\partial x \partial y} + 2 \frac{\partial^2 f_5}{\partial u \partial y} \frac{\partial u}{\partial x} + 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial f_5}{\partial u} + 2 \frac{\partial^2 f_5}{\partial v \partial y} \frac{\partial v}{\partial x} \\
&+ 2 \frac{\partial f_5}{\partial v} \frac{\partial^2 v}{\partial x \partial y} + 2 \frac{\partial^2 f_5}{\partial u \partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial^2 f_5}{\partial v \partial x} \frac{\partial v}{\partial y}) (\frac{\partial x}{\partial \bar{y}}) (\frac{\partial y}{\partial \bar{y}}) \\
&+ (f_4 \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 f_5}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial f_5}{\partial u} + 2 \frac{\partial^2 f_5}{\partial u \partial y} \frac{\partial u}{\partial y} + \frac{\partial f_5}{\partial v} \frac{\partial^2 v}{\partial y^2} + 2 \frac{\partial^2 f_5}{\partial v \partial y} \frac{\partial v}{\partial y}) (\frac{\partial y}{\partial \bar{y}})^2 .
\end{aligned} \tag{4.45}$$

Using (4.41) through (4.45), the system of equations (4.40) is written in the barred variables. The first step is to search for terms with coefficients which do not appear in the original system. Since the coefficient of $\frac{\partial^2 u}{\partial x^2}$,

$$- 2 f_4 \frac{\partial x}{\partial \bar{y}} \frac{\partial y}{\partial \bar{y}} - 2 \frac{\partial x}{\partial \bar{y}} \frac{\partial y}{\partial \bar{y}} \frac{\partial f_5}{\partial u},$$

must be equal to zero, we choose $\frac{\partial x}{\partial \bar{y}} = 0$. Actually we could, at this point, choose $\frac{\partial y}{\partial \bar{y}} = 0$. However, such an assumption would cause the coefficient of $\frac{\partial^2 u}{\partial y^2}$ to be zero. To achieve conformal invariance all other coefficients of terms in the differential equation would be required to be identically zero, but this requirement would cause unnecessary restriction. Since the coefficient of $\frac{\partial^2 v}{\partial y^2}$, $\frac{\partial f_5}{\partial v} (\frac{\partial y}{\partial \bar{y}})^2$, must be zero, we choose $\frac{\partial f_5}{\partial v} = 0$. With these assumptions the coefficient of $\frac{\partial v}{\partial y}$ in the momentum equation becomes

$$f_5 \frac{\partial f_5}{\partial y} \frac{\partial y}{\partial \bar{y}}$$

which requires that $\frac{\partial f_5}{\partial y} = 0$. Finally, observing that the terms without any coefficient reduce to

$$f_5 \frac{\partial f_5}{\partial x} \frac{\partial x}{\partial \bar{x}} + u f_4 \frac{\partial x}{\partial \bar{x}} \frac{\partial f_5}{\partial x},$$

$\frac{\partial f_5}{\partial x}$ is taken to be zero to avoid f_5 equaling $- u f_4$.

If f_5 were equal to a function of the parameter a multiplied by u, nothing would be added to the group (4.41). Consequently, f_5 is taken to be zero. With the above assumptions the transformed momentum equation becomes

$$u \frac{\partial u}{\partial x}\{ f_4^{\,2} \frac{\partial x}{\partial \bar{x}} \} + \frac{\partial u}{\partial y}\{ u f_4^{\,2} \frac{\partial y}{\partial \bar{x}} + f_4 f_\eta \frac{\partial y}{\partial \bar{y}} + f_6 f_4 \frac{\partial y}{\partial \bar{y}} v$$

$$- f_4 \frac{\partial^2 y}{\partial \bar{y}^2} \} - \frac{\partial^2 u}{\partial y^2}\{ + f_4 (\frac{\partial y}{\partial \bar{y}})^2 \} - P(\bar{x}, \bar{y}) = 0. \qquad (4.46)$$

Before proceeding further with the momentum equation, the continuity equation will be considered to see if there are any additional restrictions. The transform of the continuity equation is

$$\frac{\partial u}{\partial x}\{ f_4 \frac{\partial x}{\partial \bar{x}} \} + \frac{\partial u}{\partial y}\{ f_4 \frac{\partial y}{\partial \bar{x}} + \frac{\partial f_7}{\partial u} \frac{\partial y}{\partial \bar{y}} \}$$

$$+ \frac{\partial v}{\partial y}\{ f_6 \frac{\partial y}{\partial \bar{y}} \} + \{ \frac{\partial f_7}{\partial y} \frac{\partial y}{\partial \bar{y}} \} = 0. \qquad (4.47)$$

The last term requires $\frac{\partial f_7}{\partial y} = 0$ and the second term requires

$$f_4 \frac{\partial y}{\partial \bar{x}} + \frac{\partial f_7}{\partial u} \frac{\partial y}{\partial \bar{y}} = 0. \qquad (4.48)$$

To satisfy the condition of conformal invariance for the continuity equation, it is necessary to require

$$f_4 \frac{\partial x}{\partial \bar{x}} = f_6 \frac{\partial y}{\partial \bar{y}} . \qquad (4.49)$$

Then (4.47) becomes

$$f_6 \frac{\partial y}{\partial \bar{y}}\{ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \} = 0. \qquad (4.50)$$

Because $\frac{\partial x}{\partial \bar{y}} = 0$, $\frac{\partial y}{\partial \bar{y}}$ cannot be a function of $\bar{y}$ if (4.49) is to be satisfied. Hence, we assume that $\frac{\partial^2 y}{\partial \bar{y}^2} = 0$.

Returning to the momentum equation (4.46), we assume that

$$u\, f_4^{\,2}\ \frac{\partial y}{\partial \bar{x}} + f_4\, f_7\ \frac{\partial y}{\partial \bar{y}} = 0 \tag{4.51}$$

in order to eliminate terms which do not appear in the original momentum equation. To satisfy (4.51) we set $f_7 = p_1(a)\ h_1(x)\ u$. Substituting into (4.51) furnishes

$$f_4\ \frac{\partial y}{\partial \bar{x}} + p_1(a)\ h_1(x)\ \frac{\partial y}{\partial \bar{y}} = 0. \tag{4.52}$$

Finally, to satisfy the condition of conformal invariance we let

$$f_4^{\,2}\ \frac{\partial x}{\partial \bar{x}} = f_6\ f_4\ \frac{\partial y}{\partial \bar{y}} = f_4\ \left(\frac{\partial y}{\partial \bar{y}}\right)^2 = f_9(a) \tag{4.53}$$

and

$$P(\bar{x}, \bar{y}) = f_9(a)\ P(x, y). \tag{4.54}$$

Thus carrying an unknown function can be accomplished by simply forcing it to satisfy the condition of conformal invariance for the equation as a whole and accepting whatever functional form arises.

To satisfy the condition of conformal invariance for the system of equations (4.40), we have the following restrictions on (4.41):

$$\frac{\partial x}{\partial \bar{y}} = 0 \tag{4.55a}$$

$$f_5 = 0 \tag{4.55b}$$

$$\frac{\partial f_7}{\partial y} = 0 \tag{4.55c}$$

$$f_4 \frac{\partial x}{\partial \bar{x}} = f_6 \frac{\partial y}{\partial \bar{y}} \tag{4.55d}$$

$$f_4 \frac{\partial y}{\partial \bar{x}} + p_1(a)\, h_1(x) \frac{\partial y}{\partial \bar{y}} = 0 \tag{4.55e}$$

$$f_6 \frac{\partial y}{\partial \bar{y}} = \left(\frac{\partial y}{\partial \bar{y}}\right)^2 . \tag{4.55f}$$

The system of (4.55) does not entirely determine the transformation (4.41). Equation (4.55f) implies that $\frac{\partial y}{\partial \bar{y}} = f_6$ and (4.55d) implies that $\frac{\partial x}{\partial \bar{x}} = \frac{f_6^{\,2}}{f_4}$. Several cases must be examined.

Case I. $h_1(x) = 0$.

From (4.55e) $\frac{\partial y}{\partial \bar{x}} = 0$, whereupon the group (4.41) becomes

$$\begin{aligned} \bar{u} &= f_4(a)\, u \\ \bar{v} &= f_6(a)\, v \\ \bar{x} &= \frac{f_4}{f_6^{\,2}}\, x \\ \bar{y} &= \frac{1}{f_6}\, y \end{aligned} \tag{4.56}$$

As long as there exists an a^o such that $f_4(a^o) = f_6(a^o) = 1$, then (4.56) meets the requirements of a group. To generate the invariants of (4.56) let $f_4 = f_6^{\,m}$ where m is a constant. The invariants may be found by forming simple ratios and are

$$\lambda_1 = \frac{y}{(x)^{-\frac{1}{m-2}}}$$

$$\lambda_2 = u\, y^m \tag{4.57}$$

$$\lambda_3 = y\, v$$

If $m = 0$ the invariants reduce to the classical ones used for the case of a flat plate.

Case II. $h_1 = 1$.

Equation (4.55)e is solved to give

$$\frac{\partial y}{\partial \bar{x}} = - p_1(a)\, \frac{f_6}{f_4}\, . \tag{4.58}$$

Upon integrating we find that

$$y = f_6 \bar{y} - p_1(a)\, \frac{f_6}{f_4}\, \bar{x}. \tag{4.59}$$

Since $\bar{x} = \frac{f_4}{f_6^{\,2}}\, x$, equation (4.59) is easily inverted and we have

$$\bar{y} = \frac{1}{f_6(a)}\, y + \frac{p_1(a)}{f_6^{\,2}(a)}\, x. \tag{4.60}$$

To satisfy the closure property there must exist an a_3 such that

$\bar{y}(\bar{x}_1, \bar{y}_1, a_2) = \bar{y}(x, y, a_3)$. With (4.60) we require that

$$\frac{1}{f_6(a_2)} \left[\frac{1}{f_6(a_1)}\, y + \frac{p_1(a_1)}{f_6^{\,2}(a_1)}\, x \right] + \frac{p_1(a_2)}{f_6^{\,2}(a_2)} \left[\frac{f_4(a_1)}{f_6^{\,2}(a_1)} \right] x = \frac{y}{f_6(a_3)} + \frac{p_1(a_3)}{f_6^{\,2}(a_3)}\, x. \tag{4.61}$$

As in the case of the Burgers' equation we notice that $f_6(a_3) = f_6(a_2)\ f_6(a_1)$ and that a_3 is not affected by interchanging a_1 and a_2. Hence, we write

$$\frac{p_1(a_3)}{f_6^{\ 2}(a_3)} = \frac{p_1(a_1)f_6(a_2)}{f_6^{\ 2}(a_1)f_6^{\ 2}(a_2)} + \frac{p_1(a_2)f_4(a_1)}{f_6^{\ 2}(a_2)f_6^{\ 2}(a_1)}$$

$$= \frac{p_1(a_2)f_6(a_1)}{f_6^{\ 2}(a_2)f_6^{\ 2}(a_2)} + \frac{p_1(a_1)f_4(a_2)}{f_6^{\ 2}(a_2)f_6^{\ 2}(a_1)}$$

or

$$p_1(a_1)\ [f_6(a_2) - f_4(a_2)] = p_1(a_2)\ [f_6(a_1) - f_4(a_1)].$$

The above relation implies that $p_1(a) = [f_6(a) - f_4(a)]$ and the group of continuous transformations becomes

$$\begin{aligned} \bar{x} &= \frac{f_4}{f_6^{\ 2}}\,x \\ \bar{y} &= \frac{1}{f_6}\,y + \frac{x}{f_6^{\ 2}}\,(f_6 - f_4) \\ \bar{u} &= f_4\ u \\ \bar{v} &= f_6\ v + (f_6 - f_4)u \end{aligned} \tag{4.62}$$

Letting a° denote the identity element for the group, the invariants are found by solving

$$\left.\frac{\partial\bar{x}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial\lambda_i}{\partial x} + \left.\frac{\partial\bar{y}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial\lambda_i}{\partial y} + \left.\frac{\partial\bar{u}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial\lambda_i}{\partial u} + \left.\frac{\partial\bar{v}}{\partial a}\right|_{a=a^{\circ}} \frac{\partial\lambda_i}{\partial v} = 0. \tag{4.63}$$

Since there exists one invariant of the subgroup $\bar{x}$ and $\bar{y}$, we choose $\lambda_1 = \lambda_1(x,\ y)$ and obtain

$$(f_4'(a^o) - 2f_6'(a^o))x\frac{\partial\lambda_1}{\partial x} + [-f_6'(a^o)y + (f_6'(a^o)-f_4'(a^o))x]\frac{\partial\lambda_1}{\partial y} = 0. \tag{4.64}$$

The solution to (4.64) can be found by the method of Lagrange and is

$$\lambda_1 = y\left[x\right]^{\frac{f_6'(a^o)}{(f_4'(a^o)-2f_6'(a^o))}} + \left[x\right]^{\frac{f_4'(a^o)-f_6'(a^o)}{(f_4'(a^o)-2f_6'(a^o))}}. \tag{4.65}$$

To check whether λ_1 is an invariant, form (4.65) in the barred variables and use the group (4.62) to obtain

$$y\left[\bar{x}\right]^{\frac{f_6'}{(f_4'-2f_6')}} + \left[\bar{x}\right]^{\frac{f_4'-f_6'}{f_4'-2f_6'}} =$$

$$\frac{1}{f_6}y + \frac{x}{f_6^2}(f_6-f_4) \quad \frac{f_4}{f_6^2}\left[x\right]^{\frac{f_6'}{(f_4'-2f_6')}} + \frac{f_4}{f_6^2}\left[x\right]^{\frac{f_4'-f_6'}{f_4'-2f_6'}} \tag{4.66}$$

where f_4' and f_6' are evaluated at $a = a^o$. Expanding (4.66) we find

$$\lambda_1 = \frac{1}{f_6}\left[\frac{f_4}{f_6^2}\right]^{\frac{f_6'}{(f_4'-2f_6')}} y\left[x\right]^{\frac{f_6'}{f_4'-2f_6'}} + \frac{f_6-f_4}{f_6^2}\left[\frac{f_4}{f_6^2}\right]^{\frac{f_6'}{(f_4'-2f_6')}} \left[x\right]^{\frac{f_6'}{f_4'-2f_6'}+1}$$

$$+\left[\frac{f_4}{f_6^2}\right]^{\frac{f_4'-f_6'}{f_4'-2f_6'}} \left[x\right]^{\frac{f_4'-f_6'}{f_4'-2f_6'}} \tag{4.67}$$

Requiring equality between (4.67) and (4.65), forces

$$\frac{1}{f_6}\left[\frac{f_4}{f_6^{\,2}}\right]^{\frac{f_6'}{(f_4'-2f_6')}} = 1 \tag{4.68}$$

and

$$\frac{f_6-f_4}{f_6^{\,2}}\left[\frac{f_4}{f_6^{\,2}}\right]^{\frac{f_6'}{(f_4'-2f_6')}} + \left[\frac{f_4}{f_6^{\,2}}\right]^{\frac{f_4'-f_6'}{f_4'-2f_6'}} = 1. \tag{4.69}$$

Equations (4.68) and (4.69) are satisfied if $f_4 = (f_6)^m$ where m is an arbitrary constant. Our earlier choice is thus justified.

A second invariant may be found by choosing $\lambda_2 = \lambda_2(x, y, u)$ and solving equation (4.63). The result is

$$H_1(\lambda_1, u\, x^{-\frac{m}{m-2}}) = 0 \tag{4.70}$$

where H_1 is an arbitrary function.

The third and last invariant of the group (4.62) is found by choosing $\lambda_3 = \lambda_3(u, v, x, y)$ and again solving (4.63). The result is

$$H_2(\lambda_1, u\, x^{-\frac{m}{m-2}}, v\, u^{-\frac{1}{m}} + u^{\frac{m-1}{m}}) = 0 \tag{4.71}$$

where H_2 is an arbitrary function. The functions H_1 and H_2 implicitly define a set of similarity variables for the boundary layer equations. A special choice of these functions is

$$\lambda_1 = y\, x^{\frac{1}{m-2}} + x^{\frac{m-1}{m-2}} \tag{4.72}$$

$$\lambda_2 = \lambda_2(\lambda_1) = u\, x^{-\frac{m}{m-2}} \tag{4.73}$$

$$\lambda_3 = \lambda_3(\lambda_1) = v\, u^{-\frac{1}{m}} + u^{\frac{m-1}{m}} . \tag{4.74}$$

In addition to the above transformation, a transformation consistent with the condition of conformal invariance must be developed for $P(x, y)$. Differentiating (4.54) with respect to the parameter $\underline{a}$ produces

$$\frac{\partial P}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial a} + \frac{\partial P}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial a} = \frac{\partial}{\partial a} (f_9(a))\, P(x, y).$$

Since $f_9(a) = f_4\, f_6{}^2 = f_6{}^{m+2}$, the above equation becomes

$$\frac{\partial \bar{x}}{\partial a} \frac{\partial P}{\partial \bar{x}} + \frac{\partial P}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial a} = (m + 2)\, f_6{}^{m+1}\, f_6{}'\, P(x, y)$$

and setting $a = a_o$ we have

$$\left.\frac{\partial \bar{x}}{\partial a}\right|_{a=a_o} \frac{\partial P}{\partial x} + \left.\frac{\partial \bar{y}}{\partial a}\right|_{a=a_o} \frac{\partial P}{\partial y} = (m + 2)\, f_6{}'(a_o)\, P(x, y) \tag{4.75}$$

According to theorem (2.1) of Chapter II, (4.75) represents a necessary and sufficient condition for $P(x, y)$ to be conformally invariant under the group (4.62). Substituting the appropriate derivatives from (4.62), (4.75) yields

$$f_6{}'(a_o)\,(m - 2)\, x \frac{\partial P}{\partial x} + f_6{}'(a_o)\,[-y+(1-m)x] \frac{\partial P}{\partial y} = (m+2)\, f_6{}'(a_o)\, P(x, y)$$

or

$$(m - 2)x \frac{\partial P}{\partial x} + [- y + (1 - m) x] \frac{\partial P}{\partial y} = (m + 2) P(x, y) \qquad (4.76)$$

The solution of (4.76) is found by the method of Lagrange to be

$$H_3(\lambda_1, P x^{-\frac{m+2}{m-2}}) = 0 \qquad (4.77)$$

where H_3 is an arbitrary function. Equation (4.77) implicitly specifies the functional form of P(x, y) which can be retained in the momentum equation without destroying the similarity generated by group (4.61).

A special choice of (4.77) is

$$P = x^{\frac{m+2}{m-2}} H_4(\lambda_1) \ . \qquad (4.78)$$

If it is desired that P be a function of x only, we just need to choose H_4 to be constant.

Substituting (4.72), (4.73), (4.74) and (4.78) into (4.40a) and (4.40b) gives

$$H_4(\lambda_1) + \lambda_2'' = \frac{\lambda_2\lambda_2'}{m-2} (\lambda_1) + \frac{m}{m-2} \lambda_2^2 + \lambda_2' \lambda_3 \lambda_2^{\frac{1}{m}} \qquad (4.79)$$

$$\frac{\lambda_2'\lambda_1}{m-2} + \frac{m}{m-2} \lambda_2 + \lambda_2^{\frac{1}{m}} [\frac{1}{m} \lambda_3 \lambda_2^{-1} \lambda_2' + \lambda_3^1] = 0 \qquad (4.80)$$

Equations (4.79) and (4.80) are ordinary differential equations with λ_1 as the independent variable and λ_2 and λ_3 as the dependent variables.

Case III. $h_1(x) = x$.

Substituting $h_1(x)$ into (4.55e), we obtain

$$\frac{\partial y}{\partial \bar{x}} = - x \frac{p_1}{f_4} \frac{\partial y}{\partial \bar{y}} = - x \frac{p_1}{f_4} f_6 . \tag{4.81}$$

As in Cases I and II, $\bar{x} = \frac{f_4}{f_6^{\,2}} x$, and (4.81) may be integrated to obtain

$$y = f_6 \bar{y} - p_1 \frac{f_6^{\,3}}{f_4^{\,2}} \frac{(\bar{x})^2}{2} . \tag{4.82}$$

Inverting (4.82), the set of transformations gives

$$\begin{aligned} \bar{x} &= \frac{f_4}{f_6^{\,2}} x \\ \bar{u} &= f_4 u \\ \bar{v} &= f_6 v + x u p_1 \\ \bar{y} &= \frac{1}{f_6} y + \frac{p_1}{2 f_6^{\,2}} x^2 . \end{aligned} \tag{4.83}$$

The procedure for obtaining p_1, such that (4.83) satisfies the closure property, is the same as in Case I, whence p_1 is found to be

$$p_1(a) = f_6(a) - \frac{f_4^{\,2}(a)}{f_6^{\,2}(a)} . \tag{4.84}$$

Substituting (4.84) into (4.83) yields

$$\bar{x} = \frac{f_4}{f_6{}^2}\ x$$

$$\bar{y} = \frac{1}{f_6}\ y + \left(\frac{1}{2f_6} - \frac{f_4{}^2}{2f_6{}^4}\right) x^2$$

$$\bar{v} = f_6\ v + x\ u\ [f_6 - \frac{f_4{}^2}{f_6{}^2}] \tag{4.85}$$

$$\bar{u} = f_4\ u.$$

The invariants of (4.85) must satisfy

$$\left.\frac{\partial \bar{x}}{\partial a}\right|_{a=a_o} \frac{\partial \lambda_i}{\partial x} + \left.\frac{\partial \bar{y}}{\partial a}\right|_{a=a_o} \frac{\partial \lambda_i}{\partial y} + \left.\frac{\partial \bar{u}}{\partial a}\right|_{a=a_o} \frac{\partial \lambda_i}{\partial u} + \left.\frac{\partial \bar{v}}{\partial a}\right|_{a=a_o} \frac{\partial \lambda_i}{\partial v} = 0$$

or

$$(f_4{}' - 2f_6{}')\ x\ \frac{\partial \lambda_i}{\partial x} + [-f_6{}'y + \tfrac{1}{2}\ x^2\ (3f_6{}' - 2f_4{}')]\ \frac{\partial \lambda_i}{\partial y} + f_4{}'\ u \frac{\partial \lambda_i}{\partial u}$$

$$+ [f_6{}'\ v + x\ u(3f_6{}' - 2f_4{}')]\ \frac{\partial \lambda_i}{\partial v} = 0. \tag{4.86}$$

Letting $\lambda_1 = \lambda_1(x, y)$ we find that

$$\lambda_1 = y[x]^{\frac{f_6{}'}{(f_4{}'-2f_6{}')}} + \tfrac{1}{2}[x]^{\left(\frac{2f_4{}'-3f_6{}'}{f_4{}'-2f_6{}'}\right)} \tag{4.87}$$

where $f_4{}'$ and $f_6{}'$ are evaluated at the identity element a_o. As in Case II the requirement that λ_1 be absolutely invariant under (4.85) forces $f_4(a) = [f_6(a)]^m$ and (4.87) becomes

$$\lambda_1 = y\ x^{\frac{1}{m-2}} + \tfrac{1}{2}\ x^{\frac{2m-3}{m-2}}\ . \tag{4.88}$$

Following the same procedures as in Case II, two other particular invariants are found to be

$$\lambda_3 = \lambda_3(\lambda_1) = v\,u^{-\frac{1}{m}} + x\,u^{1-\frac{1}{m}} \tag{4.89}$$

$$\lambda_2 = \lambda_2(\lambda_1) = u\,x^{-\frac{m}{m-2}}. \tag{4.90}$$

Again as in Case II the functional form of $P(x, y)$ is determined by solutions of

$$(m - 2)\,x\,\frac{\partial P}{\partial x} + [-\,y + \tfrac{1}{2}\,x^2(3 - 2m)]\,\frac{\partial P}{\partial y} = (m + 2)\,P(x, y) \tag{4.91}$$

and a special form of $P(x, y)$ is

$$P = x^{\frac{m+2}{m-2}}\,H_5(\lambda_1), \tag{4.92}$$

where H_5 is an arbitrary function of λ_1. If P is required to be a function of x only, then H_5 may be taken to be a constant.

Substitution of (4.89), (4.90), and (4.92) into equations (4.40a) and (4.40b) furnishes

$$H_5(\lambda_1) + \lambda_2'' = \frac{\lambda_2\lambda_2'}{m-2}\,\lambda_1 + \frac{m}{m-2}\,\lambda_2^{\;2} + \lambda_2'\,\lambda_3\,\lambda_2^{\frac{1}{m}} \tag{4.93}$$

$$\frac{\lambda_2'\lambda_1}{m-2} + \frac{m}{m-2}\,\lambda_2 + \lambda_2^{\frac{1}{m}}\,[\frac{\lambda_3}{m}\,\lambda_2^{-1}\,\lambda_2' + \lambda_3'] = 0 \tag{4.94}$$

It is interesting to note that equations (4.93) and (4.94) are identical to (4.79) and (4.80). Also interesting is that $\lambda_1 = 0$ in Case II implies that $y = -\,x$ and $\lambda_1 = 0$ in Case III implies that $y = -\,\frac{1}{2}\,x^2$.

Boundary Conditions

The boundary conditions for (4.40a) and (4.40b) for the case of a zero pressure gradient are

$$u = v = 0 \quad \text{on the surface} \tag{4.95}$$

and

$$\lim_{y \to \infty} u = 1 \tag{4.96}$$

Is the group (4.62) consistent with the above conditions? The necessary and sufficient conditions for u = 0 and v = 0 to be invariant curves are

$$\left.\frac{\partial \bar{x}}{\partial a}\right|_{a=a^{o}} \frac{\partial u}{\partial x} + \left.\frac{\partial \bar{y}}{\partial a}\right|_{a=a^{o}} \frac{\partial u}{\partial y} = \omega_1(x, y)\, u \tag{4.97}$$

$$\left.\frac{\partial \bar{x}}{\partial a}\right|_{a=a^{o}} \frac{\partial v}{\partial x} + \left.\frac{\partial \bar{y}}{\partial a}\right|_{a=a^{o}} \frac{\partial v}{\partial y} = \omega_2(x, y)\, v \tag{4.98}$$

where $\omega_1(x, y)$ and $\omega_2(x, y)$ are arbitrary functions of their arguments. Choosing $\omega_1 = mf_6'(a^o)$ and recalling that $f_4 = f_6^{m}$, (4.97) integrates to

$$H_6(u\, x^{-\frac{m}{m-2}}, \lambda_1) = 0 \tag{4.99}$$

where H_6 is an arbitrary function and λ_1 is defined by (4.72). A particular form of (4.99) is

$$u = x^{\frac{m}{m-2}}\, H_7\,(\lambda_1). \tag{4.100}$$

If the surface is described by $\lambda_1 = 0$, then choosing $H_7(0) = 0$ we conclude that the condition u = 0 on the surfaces is consistent with the group (4.62). A similar argument holds for the condition $v = 0$ on the surface.

The condition (4.96) requires that $u(x, y)$ approach a constant for large y. We address our attention to finding the nature of the family of invariant curves $u(x, y) = 1$. According to Cohen [24] the necessary and sufficient condition for $u = 1$ to be an invariant curve is

$$\frac{\partial \bar{x}}{\partial a}\bigg|_{a=a^\circ} \frac{\partial u}{\partial x} + \frac{\partial \bar{y}}{\partial a}\bigg|_{a=a^\circ} \frac{\partial u}{\partial y} = F_1(u) \tag{4.101}$$

where $F_1(u)$ is an arbitrary function. Following the method of Lagrange the solution to (4.101) is

$$I(u) = \ln x^{\frac{1}{m-2}} H_8(\lambda_1) \tag{4.102}$$

where $I(u) = \int \frac{du}{F_1(u)}$ and H_8 is an arbitrary function. If $\bar{u}$ is to be constant when $u = 1$, then the right hand side of (4.102) must be a constant under the group (4.62). Unless x is taken to be constant this condition is not satisfied and we are forced to search for a restriction. If $F_1(u)$ is taken to be zero, the preceding analysis is invalid. In that case a particular solution to (4.101) becomes

$$u = H_9(\lambda_1), \tag{4.103}$$

where H_9 is an arbitrary function. Equation (4.103) implies that $\bar{u} = u$ and that $\bar{u} = 1$ when $u = 1$; hence, the boundary condition (4.96) may be satisfied if $\bar{u} = u$ in the group (4.62). Consequently, we may take $f_4 = 1$ and derive a new set of invariants which will be compatible with the boundary conditions (4.95) and (4.96).

On the other hand if the boundary condition (4.96) is

$$\lim_{y \to \infty} ux^{-\frac{m}{m-2}} = 1, \tag{4.104}$$

(4.100) confirms that $\bar{u}\,\bar{x}^{-\frac{m}{m-2}}$ is constant whenever $u\,x^{-\frac{m}{m-2}}$ is and no modification of the group is necessary. Equation (4.104) implies that the free stream velocity is

$$U_\infty = U_c\, x^{\frac{m}{m-2}} \tag{4.105}$$

and the pressure gradient is

$$\frac{U_\infty}{U_c^{\,2}} \frac{dU_\infty}{dx} = x^{\frac{m+2}{m-2}}. \tag{4.106}$$

Equation (4.106) meets the requirement set forth in (4.78).

CHAPTER V

APPLICATION OF THE INFINITESIMAL GROUP METHOD

Burgers' Equation

For the Burgers' equation $u_t + uu_x = u_{xx}$ we shall seek an infinitesimal transformation of the form

$$\bar{x} = x + X(x, t, u) + O(\varepsilon^2)$$

$$\bar{t} = t + \varepsilon + O(\varepsilon^2) \tag{5.1}$$

$$\bar{u} = u + \varepsilon U(x, t, u) + O(\varepsilon^2)$$

such that

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} - \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \omega_3(x, t, u) \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2}\right]. \tag{5.2}$$

The relevant derivatives required are

$$\frac{\partial \bar{u}}{\partial \bar{x}} = u_x + \varepsilon[U_x + (U_u - X_x)u_x - X_u u_x^2] + O(\varepsilon^2) \tag{5.3}$$

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = u_{xx} + \varepsilon[U_{xx} + (2U_{xu} - X_{xx})u_x + (U_{uu} - 2X_{xu})u_x^2$$
$$-X_{uu} u_x^3 + (U_u - 2X_x)u_{xx} - 3X_u u_{xx} u_x] + O(\varepsilon^2) \tag{5.4}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} = u_t + \varepsilon[U_t + (U_u)u_t - X_t u_x - X_u u_x u_t] + O(\varepsilon^2) \tag{5.5}$$

Following the requirements outlined in Chapter III we let the condition

$$X u_x + u_t = U \tag{5.6}$$

determine the similarity variables and also use it in satisfying (5.2). Using (5.6) and the Burgers' equation we see that

$$u_{xx} = u_t + uu_x = U + (u - X)u_x. \tag{5.7}$$

Using (5.3) through (5.7), we write

$$\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial u}{\partial t} = \varepsilon\{U_t + UU_u + u_x(-XU_u - X_t - X_uU) + (u_x)^2 \ (X_uX)\} + O(\varepsilon^2) \tag{5.8}$$

$$\bar{u}\,\frac{\partial \bar{u}}{\partial \bar{x}} - u\,\frac{\partial u}{\partial x} = \varepsilon\{u\,U_x + u_x(U + uU_u - uX_x) + (u_x)^2(-X_uu)\} + O(\varepsilon^2) \tag{5.9}$$

$$\begin{aligned} -\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - \frac{\partial^2 u}{\partial x^2}\right) = \varepsilon\{ &- U_{xx} - U(U_u - 2X_x) + u_x\,[2U_{xu} \\ &- X_{xx} - (u - X)\ (U_u - 2X_x) + 3X_uU] \\ &+ (u_x)^2\ [-U_{uu} + 2X_{xu} + 3X_u(u - X)] \\ &+ (u_x)^3\ [X_{uu}]\ \} + O(\varepsilon^2). \end{aligned} \tag{5.10}$$

To satisfy (5.2), the sum of the right-hand sides of (5.8), (5.9), and (5.10) must vanish. As u_x or powers of u_x are not identically zero, we require that the coefficients of $(u_x)^o$, u_x, $(u_x)^2$, and $(u_x)^3$ in the sum mentioned above vanish.

Setting the sum of the coefficients of $(u_x)^3$ equal to zero gives

$$X_{uu} = 0. \tag{5.11}$$

Similarly, from the coefficients of $(u_x)^2$, (u_x), and $(u_x)^o$ we find that

$$U_{uu} = + 2X_{xu} + 2u\, X_u - 2X_u X \tag{5.12}$$

$$U - X_t - (U_{xu} - X_{xx}) + uX_x - 2XX_x = 0 \tag{5.13}$$

$$U_t + u\, U_x - U_{xx} + 2UX_x = 0. \tag{5.14}$$

The solution of (5.11) is

$$X = C_2(x, t)\, u + C_1(x, t). \tag{5.15}$$

With (5.15) the integration of (5.12) produces the term $(C_2 - C_2^{\,2}) \frac{u^3}{6}$. In that case (5.13) would contain only one cubic term in u which forces either $C_2 = 1$ or $C_2 = 0$. The case $C_2 = 1$ will be considered later. Assuming that $C_2 = 0$ and integrating (5.12), we have

$$U = B(x, t)u + D(x, t). \tag{5.16}$$

Letting $C_1(x, t) = A(x, t)$ and substituting (5.16) into (5.13) and (5.14) gives

$$B\, u + D - A_t - (B_x - A_{xx}) + A_x u - 2AA_x = 0 \tag{5.17}$$

$$\begin{aligned} B_t\, u + D_t + u(B_x\, u + D_x) - [B_{xx}\, u + D_{xx}] \\ + 2(B\, u + D)\,(A_x) = 0. \end{aligned} \tag{5.18}$$

Equating coefficients of u in (5.17) to zero requires

$$B = - A_x. \tag{5.19}$$

Equating the coefficient of u^2 in (5.18) to zero requires

$$B_x = 0. \qquad (5.20)$$

Since (5.20) requires that B be only a function of t, (5.19) integrates to

$$A = -B(t)x + E(t). \qquad (5.21)$$

Equating the coefficient of u to zero in (5.18) gives

$$B_t + D_x + 2A_x B = 0 \qquad (5.22)$$

which implies that D_x is only a function of t. Hence, we have

$$D = F(t)\,x + G(t). \qquad (5.23)$$

The coefficient of u^o in (5.18) must vanish. Thus

$$D_t + 2\,D\,A_x = 0 \qquad (5.24)$$

and the vanishing of the coefficient of $(u)^o$ in (5.17) implies that

$$D - A_t - 2\,A\,A_x = 0. \qquad (5.25)$$

Substituting (5.21) and (5.23) into (5.22), (5.24), and (5.25) yields

$$F\,x + G - (-B_t\,x + E_t) + 2(-B\,x + E)\,(B) = 0 \qquad (5.26)$$

$$F_t\,x + G_t - 2(F\,x + G)\,(B) = 0. \qquad (5.27)$$

Requiring the coefficients of x^2, x, and x^o to vanish in (5.26) and (5.27), produces

$$F + B_t - 2B^2 = 0 \qquad (5.28)$$

$$G - E_t - 2BE = 0 \qquad (5.59)$$

$$F_t - 2FB = 0 \tag{5.30}$$

$$G_t - 2GB = 0. \tag{5.31}$$

The last two equations may be solved for B to obtain

$$B = \frac{F_t}{2F} = \frac{G_t}{2G}. \tag{5.32}$$

Equation (5.32) integrates to

$$F = c_1 G \tag{5.33}$$

where c_1 is a constant. Substituting (5.32) into (5.28) gives

$$c_1 G + B_t - 2B^2 = 0. \tag{5.34}$$

To satisfy the condition of conformal invariance, solutions must be obtained to the system

$$c_1 G + B_t - 2B^2 = 0 \tag{5.35}$$

$$G - E_t + 2BE = 0 \tag{5.36}$$

$$G_t - 2BG = 0 \tag{5.37}$$

and

$$F = c_1 G.$$

Case I. E = 0.

Assuming $E = 0$, (5.36) requires that $G = 0$ and consequently $F = 0$. Equation (5.35) becomes

$$B_t - 2B^2 = 0 \tag{5.38}$$

which can be integrated by elementary methods to give

$$B = -\frac{1}{(2t+M)}. \tag{5.39}$$

With this result (5.21) requires that

$$A = \frac{1}{2t+M}x = X(x, t) \tag{5.40}$$

and (5.16) yields

$$U(x, t, u) = -\frac{1}{2t+M}u. \tag{5.41}$$

Substituting (5.40) and (5.41) into (5.6) produces

$$\frac{x}{2t+M}u_x + u_t = -\frac{1}{2t+M}u. \tag{5.42}$$

The solution of the homogeneous portion of (5.42) produces the invariant of the subgroup $\bar{x}$ and $\bar{t}$ in (5.1) and the general solution of (5.42) produces an invariant of the group (5.1). These solutions are easily obtained by the method of Lagrange as

$$\eta = I_1\left(\frac{x}{(2t+M)^{\frac{1}{2}}}\right) \tag{5.43}$$

$$I_2(u\ (2t + M)^{\frac{1}{2}}, \eta) = 0, \tag{5.44}$$

where I_1 and I_2 are arbitrary functions of the indicated arguments.

A special choice of (5.43) and (5.44) is

$$\eta = \frac{x}{(2t+M)^{\frac{1}{2}}} \tag{5.45}$$

$$u = (2t + M)^{-\frac{1}{2}} F_1(\eta). \tag{5.46}$$

If the above two equations are substituted into the Burgers' equation, we obtain

$$F_1'' - F_1' F_1 + F_1' \eta + F_1 = 0. \tag{5.47}$$

With $F_1 = \frac{1}{2} F_2$, (5.47) becomes

$$F_2'' - \frac{1}{2} F_2' F_2 + F_2' \eta + F_2 = 0. \tag{5.48}$$

Equation (5.48) may be solved by the same method employed to solve (4.31) in Chapter IV.

Case II. $E = - c_2 B$.

Substituting $E = - c_2 B$ in (5.36) and multiplying the result by c_1 yields

$$c_1 G + c_1 c_2 B_t - 2c_1 c_2 B^2 = 0.$$

Assuming that $c_1 c_2 = 1$ reduces this to

$$c_1 G + B_t - 2B^2 = 0. \tag{5.49}$$

Noting that (5.49) is identical to (5.35), we need only solve the system (5.35) and (5.37). Equation (5.37) may be solved for B and the result substituted into (5.35) to obtain

$$G G_{tt} - 2(G_t)^2 + 2c_1(G^3) = 0. \tag{5.50}$$

If G_t is defined as a function of G, say $G_t = r(G)$, (5.50) may be transformed to

$$G r'r - 2r^2 G + 2c_1 G^3 = 0$$

or

$$r'r - 2r^2 + 2c_1 G^2 = 0. \tag{5.51}$$

Next define $r(G) = G^{3/2}\, y_1(x_1)$ and $x_1 = \log G$ and substitute into (5.51) to obtain

$$y_1\, y_1' - \tfrac{1}{2} y_1^2 + 2c_1 = 0. \qquad (5.52)$$

Equation (5.52) is easily solved by separation of variables and we find

$$- \tfrac{1}{2} y_1^{\,2} + 2c_1 = c_3 G, \qquad (5.53)$$

where c_3 is a constant. Since $r = G^{3/2}\, y_1(x_1)$, (5.53) is integrated to

$$G = \frac{2c_1}{(2c_1^{\,2}(t+d)+c_3)}, \qquad (5.54)$$

where d is an integration constant.

Putting (5.54) into (5.37) we obtain

$$B = - 2c_1^{\,2}\, [2c_1^{\,2}\, (t + d)^2 + c_3]^{-1}\, (t + d). \qquad (5.55)$$

Also

$$E = + 2c_1[2c_1^{\,2}\, (t+d)^2 + c_3)]^{-1}\, (t + d) \qquad (5.56)$$

$$F = + 2c_1^{\,2}[2c_1^{\,2}\, (t + d)^2 + c_3]^{-1}. \qquad (5.57)$$

Recalling that $B = + \frac{G_t}{2G}$, we have

$$X = - \frac{G_t}{2G}\, x - c_2\, \frac{G_t}{2G}\,. \qquad (5.57)$$

The Lagrange subsystem for (5.6) is

$$\frac{dx}{X} = \frac{dt}{I} = \frac{du}{U}\,. \qquad (5.58)$$

Using (5.57), the first pair of (5.58) gives

$$\frac{dx}{\frac{-G_t}{2G}(x+c_2)} = \frac{dt}{1}$$

which is integrated to

$$\ln(x + c_2) = -\tfrac{1}{2} \ln G - \ln \eta_1, \tag{5.59}$$

where η_1 is an integration constant. Using (5.54), (5.59) becomes

$$\eta_1 = \frac{1}{\sqrt{2c_1^{\,2}}} \; \frac{[2c_1^{\,2}(t+d)^2+c_3]^{\frac{1}{2}}}{x + c_2}. \tag{5.60}$$

Recalling that D = F x + G, we have U = B u + D and using (5.54), (5.55), and (5.57), we can write

$$U = \frac{G_t}{2G} u + (c_1 \; x + 1) \; (2c_1) \; [2c_1^{\,2} \; (t + d) + c_3]^{-1}. \tag{5.61}$$

To eliminate the explicit appearance of x in (5.61), (5.60) is solved for x and the relation $c_2 \; c_1 = 1$ is used giving the result

$$U = \frac{G_t}{2G} u + \sqrt{2c_1^{\,2}} \; (2c_1^{\,2} \; (t + d) + c_3)^{-\frac{1}{2}}/\eta_1. \tag{5.62}$$

Substituting (5.54) into the above produces

$$U = -\, 2c_1^{\,2} \; [2c_1^{\,2} \; (t + d) + c_3]^{-1} \; (t + d)u + \frac{\sqrt{2c_1^{\,2}} \; (2c_1^{\,2}(t+d) + c_3)^{-\frac{1}{2}}}{\eta_1}. \tag{5.63}$$

To obtain another similarity variable we must solve the second pair of equations (5.58) with (5.63). Assuming $c_3 = 0$ we have

$$\frac{du}{dt} = -\,(t + d)^{-1} \; u + {}^{1}/\eta_1 \; (t + d)^{-1}. \tag{5.64}$$

This differential equation is easily integrated by elementary methods and has the following particular solution:

$$F_3 = (t + d)\ (u - {}^{1}/\eta_1), \tag{5.65}$$

where F_3 is an undetermined function of η_1. It should be emphasized that, in the integration of (5.64), η_1 is held constant.

With $c_3 = 0$, (5.60) becomes

$$\eta_1 = \frac{t+d}{x+c_2}\ . \tag{5.66}$$

Substituting (5.66) and (5.65) into the Burgers' equation gives

$$- F_3\ F_3' = 2\eta_1\ F_3' + \eta_1^{\ 2}\ F_3''. \tag{5.67}$$

or

$$\frac{d}{d\eta}\,(\eta_1^2\ F_3' + \tfrac{1}{2}\ F_3^{\ 2}) = 0. \tag{5.68}$$

Integrating (5.68) produces

$$\eta_1^{\ 2}\ F_3' + \tfrac{1}{2}\ F_3^{\ 2} = c_4 \tag{5.69}$$

and integrating again gives

$$\frac{2}{c_5}\ \tanh^{-1}\ \left(\frac{F_3}{c_5}\right) = -\ \eta_1^{\ -1} + c_6, \tag{5.70}$$

where $c_5 = \quad 2c_4$. Rearranging (5.70) produces the closed form solution to (5.67)

$$F_3 = c_5 \tanh\ \left[\left(\frac{c_5}{2}\right)\ \left(c_6 - \frac{1}{\eta_1}\right)\right] \tag{5.71}$$

Substituting the definitions of F_3 and η_1 into (5.71) gives the following closed form solution to the Burgers' equation:

$$u = c_5\ (t + d)^{-1}\ \tanh\ [\frac{c_5}{2}\ (c_4 - \frac{x+c_2}{t + d})] + \frac{x+c_2}{t + d}\ . \qquad (5.71a)$$

Case III. E Unspecified.

Equation (5.30) may be solved for B and substituted into (5.28) to obtain

$$2F^3 + F\ F_{tt} - 2(F_t)^2 = 0. \qquad (5.72)$$

Equation (5.72) has a solution analogous to that of (5.49) with $c_1 = 1$, namely

$$F = 2\ [2(t + d)^2 + c_3]^{-1}. \qquad (5.73)$$

Substituting (5.73) into (5.30) produces

$$B = -\ 2\ [2(t + d)^2 + c_3]^{-1}\ (t + d). \qquad (5.74)$$

Putting (5.74) into (5.31) and integrating gives

$$G = [2\ (t + d)^2 + c_3]^{-1}. \qquad (5.75)$$

Substituting (5.75) into (5.29) and assuming that $c_3 = 0$, we obtain

$$E_t + 2E\ (t + d)^{-1} = \tfrac{1}{2}\ (t + d)^{-2}. \qquad (5.76)$$

The above equation has the general solution

$$E = c_6\ (t + d)^{-2} + \tfrac{1}{2}\ (t + d)^{-1}. \qquad (5.77)$$

Because $X = -\ B\ x + E$, the first pair of equations in (5.58) becomes

$$\frac{dx}{dt} = (t + d)^{-1}\ (x + \tfrac{1}{2}) + c_6\ (t + d)^{-2}. \qquad (5.78)$$

Integrating we obtain

$$x = \eta_3 (t + d) - \frac{c_6}{2} (t + d)^{-1} - \tfrac{1}{2}. \tag{5.79}$$

Recalling that D = F x + G and U = B u + D, we can write

$$U = - (t + d)^{-1} u + (t + d)^{-2} (x + \tfrac{1}{2}). \tag{5.80}$$

Substituting (5.80) into the second pair of equations (5.58) and using (5.79) gives

$$\frac{du}{dt} = - (t + d)^{-1} (u - \eta_3) - \frac{c_6}{2} (t + d)^{-3}. \tag{5.81}$$

A general solution to (5.81) is

$$u = F_4 (t + d)^{-1} + \eta_3 + \frac{c_6}{2} (t + d)^{-2}, \tag{5.82}$$

where F_4 is a function of η_3.

Equations (5.80) and (5.81) define another set of similarity variables for the Burgers' equation which,when substituted into the equation,gives

$$F_4 F_4' - c_6 = F_4''. \tag{5.82a}$$

If $F_4 = - 2 F_5$, (5.82a) becomes

$$F_5'' + 2 F_5 F_5' - \frac{c_6}{2} = 0 \tag{5.83}$$

or

$$\frac{d}{d\eta_3} (F_5' + F_5^2) = \frac{c_6}{2}. \tag{5.84}$$

Integrating (5.84) gives

$$F_5' + F_5^2 = \frac{c_6}{2} \eta_3 + c_7. \tag{5.85}$$

Equation (5.85) is a special form of the Riccati equation (see Ames [33]). Accordingly, we let $F_5 = \frac{y_2'}{y_2}$ and obtain

$$y_2'' = (\frac{c_6}{2} \eta_3 + c_7)y. \tag{5.86}$$

Letting $s = \eta_3 + \frac{2c_7}{c_6}$, we can rewrite the above as

$$\frac{d^2y_2}{ds^2} - \frac{c_6}{2} s\, y_2 = 0. \tag{5.87}$$

According to Kamke [36], the solution of (5.87) is

$$y_2 = \sqrt{s}\; Z_{1/3} \left({}^{2}/3\; i \sqrt{{}^{c_6}/2}\; s^{\,3/2}\right), \tag{5.88}$$

where $Z_{1/3} = c_8 J_{1/3} + c_9 Y_{1/3}$ and $J_{1/3}$ and $Y_{1/3}$ are Bessel functions of the first and second kind. Letting $q = \sqrt{\frac{2i}{3} \frac{c_6}{2}}$, we see that

$$\frac{dy_2}{ds} = \tfrac{1}{2} (s)^{-\frac{1}{2}} Z_{1/3} (q) + s \frac{d(Z_{1/3})}{dq} \frac{dq}{ds},$$

where $\frac{dq}{ds} = i \sqrt{\frac{c_6}{2}}\; s^{\frac{1}{2}}$. Recalling that $F_5 = \frac{y_2'}{y_2}$ we obtain

$$F_5 = \frac{1}{2s} + i \frac{\sqrt{\frac{c_6}{2}}\,(s)^{\frac{1}{2}} \frac{d}{dq} (Z_{1/3})}{Z_{1/3} (q)}. \tag{5.89}$$

Equation (5.89) together with similarity variables defines another closed form solution to the Burgers' equation.

Boundary Layer Equations

The nondimensional boundary layer equations were given in Chapter IV as

$$u\, u_x + v\, u_y = P(x, y) + u_{yy}$$
$$u_x + v_y = 0. \tag{5.90}$$

To be consistent with the notation of the infinitesimal treatment of the Burgers' equation, let x be replaced by t and y by x. Then we have

$$u\, u_t + v\, u_x = P(x, t) + u_{xx} \tag{5.91}$$

$$u_t + v_x = 0. \tag{5.92}$$

An infinitesimal transformation of the form

$$\bar{x} = x + \varepsilon\, X\,(x, t)$$
$$\bar{t} = t + \varepsilon$$
$$\bar{u} = u + \varepsilon\, U(x, t, u) \tag{5.93}$$
$$\bar{v} = v + \varepsilon\, V(x, t, u, v)$$

is sought such that (5.91) and (5.92) will be conformally invariant. In addition to the derivatives given in (5.3), (5.4), and (5.5), we need the derivative:

$$\frac{\partial \bar{v}}{\partial \bar{x}} = v_x + \varepsilon\, [-v_x\, X_x + V_x + V_u\, u_x + V_v\, v_x]. \tag{5.94}$$

According to the procedure discussed in Chapter III, we require

$$X \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = U \tag{5.95}$$

$$X \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = V. \tag{5.96}$$

Using the momentum equation and (5.95) we write

$$u_{xx} = u\ (U - X\ u_x) + v\ (u_x) - P\ (x,\ y). \tag{5.97}$$

With (5.95),(5.96), and (5.97) we may write the terms appearing in (5.91) as

$$(\bar{u}\ \frac{\partial \bar{u}}{\partial \bar{t}} - u\ \frac{\partial u}{\partial t}) = \varepsilon\ \{ U\ (U - X\ u_x) + uU_t + uUU_u + uu_x\ (-X\ U_u - X_t)\ \} + 0(\varepsilon^2) \tag{5.98}$$

$$(\bar{v}\ \frac{\partial \bar{u}}{\partial \bar{x}} - v\ \frac{\partial u}{\partial x}) = \varepsilon\ \{ V\ u_x + vU_x + vu_x\ (U_u - X_x)\ \} + 0(\varepsilon^2) \tag{5.99}$$

$$- \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 u}{\partial x^2} = \varepsilon\ \{ - U_{xx} - u_x\ (2U_{xu} - X_{xx}) - (u_x)^2\ (U_{uu})$$
$$- [uU - P(x,\ t) + u_x\ (v - X_u)]\ [U_u - 2\,X_x]\ \} + 0(\varepsilon^2) \tag{5.100}$$

$$- P\ (\bar{x},\ \bar{t}) + P\ (x,\ t) = \varepsilon\ [- P_x\ X - P_t] + 0(\varepsilon^2) \tag{5.101}$$

$$(\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial u}{\partial t}) = \varepsilon\ \{ U_t + UU_u + u_x\ (-XU_u - X_t)] + 0(\varepsilon^2) \tag{5.102}$$

$$(\frac{\partial \bar{v}}{\partial \bar{x}} - \frac{\partial v}{\partial x}) = \varepsilon\ [X_x\ U + V_x - U\ V_v + u_x\ (-XX_x + V_u + U_v X)] + 0(\varepsilon^2). \tag{5.103}$$

For conformal invariance of the momentum equation the sum of the right hand side of (5.98), (5.99), (5.100), and (5.101) must vanish. Accordingly, we require the coefficients of $(u_x)^2$, (u_x), and $(u_x)^0$ to vanish in that sum. The coefficient of $(u_x)^0$ is

$$- U_{xx} - (u\,U - P)\,(U_u - 2X_x) + U^2 + uU_t + uUUu + v\,U_x - P_x\,X - P_t = 0. \qquad (5.104)$$

As vU_x is the only term containing v, we must require that $U_x = 0$. In addition we shall require P(x, t) to be conformally invariant and be determined by the relation

$$P_x\,X + P_t = P\,(U_u - 2X_x). \qquad (5.105)$$

With the foregoing observations, (5.104) reduces to

$$- uU\,(U_u - 2X_x) + U^2 + u\,U_t + uUU_u = 0. \qquad (5.106)$$

The coefficient of (u_x) is

$$- (U_{xu} - X_{xx}) - (U_u - 2X_x)\,(v - Xu) - (X\,U_u + X_t)\,u + U\,X + V + v\,(U_u - X_x) = 0. \qquad (5.107)$$

The coefficient of $(u_x)^2$ is

$$U_{uu} = 0 \qquad (5.108)$$

or

$$U = D_1\,(t)u + E_1(t). \qquad (5.109)$$

Because the continuity equation must also be conformally invariant, the sum of the right-hand sides of (5.102) and (5.103) must vanish and we require the coefficients of (u_x) and $(u_x)^0$ to vanish in that sum. The coefficient of $(u_x)^o$ is

$$U_t + UU_u + X_x U + V_x - UV_v = 0 \tag{5.110}$$

and the coefficient of (u_x) is

$$- XU_u - X_t - XX_x + V_u + V_v X = 0. \tag{5.111}$$

We shall assume that V takes the functional form

$$V = A_1(t)v + B_1\ (x, t)u + C_3\ (x, t). \tag{5.112}$$

Substituting (5.109) into (5.106) produces

$$u^2\ (D_{1t} + D_1{}^2 + 2X_x\ D_1) + u(+\ 2E_1\ D_1 + 2X_x\ E_1 + E_{1t}) + E_1^2 = 0. \tag{5.113}$$

Equating the coefficients of u^2, u, and $(u)^o$ to zero in (5.113), we have

$$D_{1t} + D_1{}^2 + 2X_x\ D_1 = 0 \tag{5.114}$$

$$E = 0. \tag{5.115}$$

As D_1 is only a function of t, (5.114) can be consistent if X_x is at most a function of t. Hence, we require

$$X = G_1(t)\ x + F_1(t). \tag{5.116}$$

With (5.109), (5.112), and (5.116), (5.107) becomes

$$2\ G_1 v - 2G_1(G_1\ x + F_1)u - (D_1\ u)\ (G_1\ x + F_1) - u(G_{1_t}\ x + F_{1_t}) + A_1 v + B_1 u + C_3 - vG_1 = 0. \tag{5.117}$$

As the above equation is almost linear in x, B_1 is required to assume the form

$$B_1 = a_1(t)\,x + b_1(t). \tag{5.118}$$

With (5.118), (5.117) is rearranged to

$$v(G_1 + A_1) + u(-2G_1^2\,x - 2G_1\,F_1 - D_1\,G_1\,x - D_1\,F_1 - G_{1t}\,x - F_{1t} + B_1)$$
$$+ C_3 = 0. \tag{5.119}$$

Equating the coefficients of v and u equal to zero and then equating the coefficients of x and x^o to zero in (5.119), we obtain

$$C_3 = 0 \tag{5.120}$$

$$+ 2G_1^2 + D_1\,G_1 + G_{1t} - a_1 = 0 \tag{5.121}$$

$$+ 2G_1\,F_1 + D_1\,F_1 + F_{1t} - b_1 = 0 \tag{5.122}$$

$$A_1 = - G_1. \tag{5.123}$$

Substituting (5.109), (5.112), and (5.116) into (5.110), we find

$$D_{1t}\,u + D_1^2\,u + G_1\,D_1\,u + a_1\,u + DGu = 0$$

or

$$D_{1t} + D_1^2 + 2G_1\,D_1 + a_1 = 0. \tag{5.124}$$

Because $X_x = G_1$, (5.114) can be consistent with (5.124) only if $a_1 = 0$. Substituting (5.109), (5.112), and (5.116) into (5.110), we find

$$x(-G_1\,D_1 - G_{1t} - 2G_1^2) + (-D_1F_1 - F_{1t} - 2G_1F_1 + b_1) = 0. \tag{5.125}$$

Equating the coefficients of x and x^o to zero in (5.125) gives

$$G_{1t} + G_1 D_1 + 2G_1^2 = 0 \tag{5.126}$$

$$F_{1t} + 2G_1 F_1 + D_1 F_1 - b_1 = 0. \tag{5.127}$$

As (5.126) and (5.127) are identical with (5.121) and (5.122) if $a_1 = 0$ and (5.114) is identical with (5.124) if $a_1 = 0$, then to satisfy conformal invariance of (5.91) and (5.92), we must solve

$$D_{1t} + D_1^2 + 2G_1 D_1 = 0 \tag{5.128}$$

$$G_{1t} + G_1 D_1 + 2G_1^2 = 0 \tag{5.129}$$

$$F_{1t} + 2G_1 F_1 + D_1 F_1 - b_1 = 0. \tag{5.130}$$

The above three equations do not uniquely determine the unknown functions F_1, D_1, G_1, and b_1. One of them may be chosen arbitrarily and conformal invariance will still be satisfied. Indeed, the arbitrariness of the above system implies that there exists an arbitrary number of similarity variables for the system (5.91) and (5.92). This fact will have a special significance in the conclusions that follow.

If we assume that $G_1 = d_1 D_1$ where d_1 is a constant, (5.128) and (5.129) collapse into the single equation

$$D_{1t} + D_1^2 (1 + 2d_1) = 0 \tag{5.131}$$

which may be integrated to

$$D_1 = \frac{1}{(1+2d_1)t}. \tag{5.132}$$

In that case

$$G_1 = \frac{d_1}{(1+2d_1)t} \tag{5.133}$$

and

$$X = G_1\ x + F_1 = \frac{d_1}{(1+2d_1)t}\ x + F_1. \tag{5.134}$$

With (5.130), (5.133), and (5.123), (5.112) becomes

$$V = -\frac{d_1}{(1+2d_1)t}\ v + (F_{1t} + \frac{2d_1}{(1+2d_1)t}\ F_1 + \frac{1}{(1+2d_1)t}\ F_1)u\ , \tag{5.135}$$

or

$$V = \frac{-d_1}{(1+2d_1)t}\ v + (F_{1t} + F_1 t^{-1})u. \tag{5.136}$$

The similarity variables may be generated by solving (5.95) and (5.96). The appropriate Lagrange subsystem is

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U} = \frac{dv}{V}. \tag{5.137}$$

Because $U = D_1 u$, the middle pair of (5.137) becomes

$$\frac{du}{u} = \frac{dt}{(1+2d_1)t} \tag{5.138}$$

which may be integrated to

$$u = t^{\frac{1}{1+2d_1}}\ \Omega, \tag{5.139}$$

where Ω is an integration constant. The remainder of the system (5.137) cannot be integrated until the function F_1 has been specified. For purposes of illustration F_1 is arbitrarily chosen as

$$F_1 = d_2 t^n. \tag{5.140}$$

With this choice and (5.139) and (5.126), another pair of the system (5.137) becomes

$$\frac{dv}{dt} = -\frac{d_1}{(1+2d_1)t}\, v + d_2(nt^{n-1} + t^{n-1})t^{\frac{1}{1+2d_1}}\, \Omega. \tag{5.141}$$

The solution to (5.141) is obtained by elementary methods and is

$$v = h\, t^{-(\frac{d_1}{1+2d_1})} + d_2\left(\frac{\frac{n+1}{1+d_1}}{n+\frac{1}{1+d_2}}\right)\, \Omega\, t^n\, t^{\frac{1}{1+2d_1}}, \tag{5.142}$$

where h is an integration constant.

The first pair of the system (5.137) becomes

$$\frac{dx}{dt} = \frac{d_1}{(1+2d_1)t}\, x + d_2\, t^n. \tag{5.143}$$

The solution of the above equation is

$$x = \zeta\, t^{\frac{d_1}{1+d_1}} + \frac{d_2\, t^{n+1}}{(n+1) - \frac{d_1}{1+2d_1}} \tag{5.144}$$

where ζ is an integration constant. According to the method of Lagrange a general solution is constructed by forming an arbitrary function of the integration constants of the subsystem. A particular choice of the functional form of the solution to (5.95) and (5.96) is obtained by choosing h and Ω to be functions of ζ. With this choice, (5.139) and (5.142) are solutions to (5.95) and (5.96) and thus describe the similarity variables for (5.91) and (5.92). Letting m be chosen

$$m = \frac{1}{1+2d_1} \tag{5.145}$$

the similarity variables become

$$\zeta = x\, t^{\frac{m-1}{2}} - b_2 t^n\, t^{\frac{m+1}{2}}, \tag{5.146}$$

$$u = t^m\, \Omega(\zeta), \tag{5.147}$$

$$v = t^{\frac{m-1}{2}}\, h(\zeta) + b_2(n+1)t^n\, t^m\, \Omega(\zeta), \tag{5.148}$$

where b_2 is defined by

$$b_2 = \frac{d_2}{(n + \frac{1+d_1}{1+2d_1})}. \tag{5.149}$$

Before introducting the similarity variables in the boundary layer equations we must determine the function P(x, y). With (5.140) and (5.134), (5.104) becomes

$$P_x\, X + P_t = P(\frac{1-2d_1}{(1+2d_1)t})$$

or

$$P_x\, X + P_t = P(2m - 1)t^{-1}. \tag{5.150}$$

The solution of (5.150) can be found by the method of Lagrange and is

$$I_3(\zeta,\ P\ t^{-(2m-1)}) = 0, \tag{5.151}$$

where I_3 is an arbitrary function. A particular choice of (5.151) is

$$P = t^{2m-1}\ I_4(\zeta). \tag{5.152}$$

Introducing the similarity variables given by (5.146), (5.147), (5.148) and (5.152) into the boundary layer equations, the following system of ordinary differential equations is obtained.

$$\Omega' \frac{(m-1)}{2} \zeta + m\Omega + h' = 0 \tag{5.153}$$

$$\Omega\Omega' \frac{(m-1)}{2} \zeta + m\Omega^2 + \Omega' h = \Omega'' + I_4. \tag{5.154}$$

If it is desired that $\frac{u^*}{U_c}$ approach t^m as x approaches infinity and P equals $\frac{U_\infty}{U_c}\ d\ (\frac{U_\infty}{U_c})/dx$, then P becomes

$$P = t^{2m-1}\ I_4 = m\ t^{2m-1} \tag{5.155}$$

from which we conclude that I_4 must be constant and equal to m.

The boundary conditions for the boundary layer problem might be

$$\lim_{x \to \infty} \frac{u^*}{U_c} = t^m \tag{5.156}$$

$$u^* = v^* = 0 \tag{5.157}$$

on the surface. Since ζ approaches infinity as x does, the first condition above is satisfied by the similarity variables if

$$\lim_{\zeta \to \infty} \Omega(\zeta) = 1. \tag{5.158}$$

The surface condition (5.157) can be satisfied if we let $\Omega(\zeta)$ and $h(\zeta)$ be zero when $\zeta = 0$. The surface corresponding to $\zeta = 0$ can be found from (5.147) to be

$$x = b_2 t^{n+1}. \tag{5.159}$$

Equations (5.153) and (5.154) were solved numerically with a Runge-Kutta routine. The resulting profile for Ω is plotted in Fig. (5.1).

There are some interesting observations concerning the similarity variables obtained here. First, with $b_2 = 0$ we have

$$\zeta = xt^{\frac{m-1}{2}} \tag{5.160}$$

$$u = t^m \Omega \tag{5.161}$$

$$v = t^{\frac{m-1}{2}} h(\zeta). \tag{5.162}$$

As b_2 does not appear in the differential equation, the solutions defining Ω and h are unchanged. If $m = 0$ the above system becomes

$$\zeta = \frac{x}{\sqrt{t}} \tag{5.163}$$

$$u = \Omega(\zeta) \tag{5.164}$$

$$v = \frac{h}{\sqrt{t}}. \tag{5.165}$$

These variables correspond to the classical boundary layer similarity variables which could be obtained with a simple finite group such as

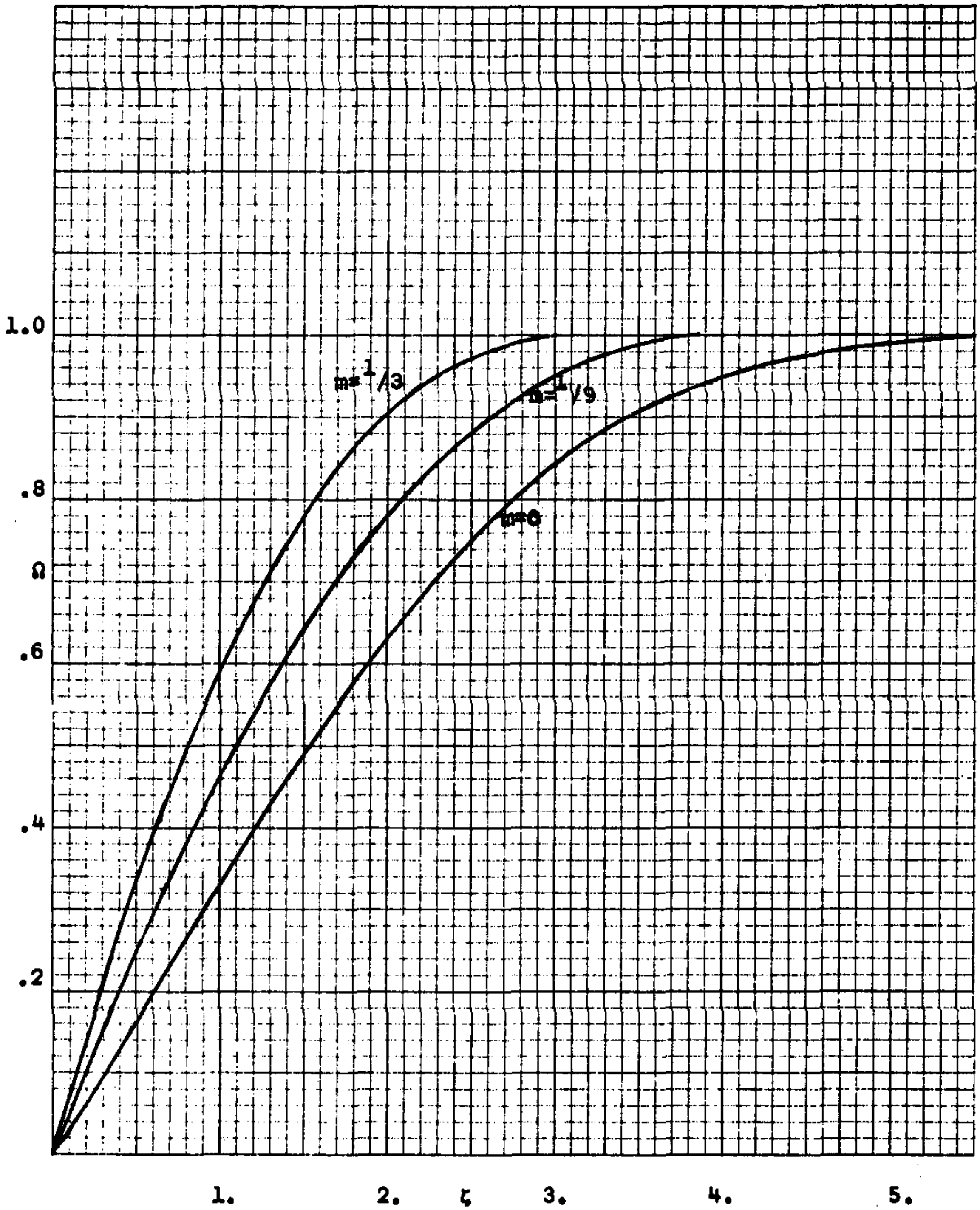

Fig. 5.1 Profile for Ω

$$\bar{x} = ax$$
$$\bar{t} = a^2 t$$
$$\bar{u} = u \qquad (5.166)$$
$$\bar{v} = av$$

Second, the appearance of t^n in two of the similarity variables is brought about by the choice of the function F_1 in (5.140). Hence, different types of surfaces can be generated by different choices of F_1.

A Shortcoming in the Method of Infinitesimal Transformations

When considering the Burgers' equation in the early part of this chapter, a special case $C_2 = 1$ was noted in connection with (5.15). Letting $C_2 = 1$ and $C_1 = 0$ reduces (5.15) to $X = u$. Equations (5.12), (5.13), and (5.14) are a set of conditions for the Burgers' equation to be conformally invariant. After some inspection with $X = u$ the only solution to these three equations is $U = 0$. According to the theory presented in Chapter III, similarity variables are solutions to

$$X \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = U. \qquad (5.167)$$

With the choices mentioned above, (5.167) becomes

$$u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0. \qquad (5.168)$$

A specific solution to (5.168) is

$$\zeta_1 = \frac{x}{u} - t \qquad (5.169)$$

$$u = \chi(\zeta_1) \qquad (5.170)$$

which should represent a set of similarity variables for the Burgers' equation. Direct substitution of (5.169) and (5.170) into the Burgers' equation produces

$$\chi'' - \chi' \left[u_x + \frac{u_x}{\chi'}\right] = 0 \qquad (5.171)$$

where

$$u_x = \chi'/[u + \frac{x}{u}].$$

Instead of the expected ordinary differential equation, the result obtained in (5.71) is a mixture - a signal that something is wrong with the procedure.

In determining conformal invariance (5.7) was used; however, in the case where U = 0 and X = u, that equation says that $u_{xx} = 0$ which is an undesirable result. Hence, it seems that using (5.61) in determing the requirements for conformal invariance may cause an undesirable result if that equation in some way contradicts the original problem.

CHAPTER VI

CONCLUSION

In Chapters IV and V similarity variables have been obtained by using finite and infinitesimal groups. The two methods differ in difficulty and in results produced. For this reason some comparisons will be made herein.

The difficulty in application of the infinitesimal method lies in the complexity of systems of nonlinear partial or ordinary differential equations which arise. This difficulty is eased by the fact that, in the problems illustrated, the unknown functions are often linear with coefficients depending upon a single variable. Equations (5.15) and (5.16) are examples of this simplification. Hence the problem is easily reduced to one of determining the solution of a simpler set of ordinary differential equations. Equations (5.28), (5.29), (5.30), and (5.31) are examples of such a set. Although, in this paper , solutions to such a system have been obtained by substituting one equation into another, experience has shown that a trial and error method using a simple function such as a polynomial is often effective because only simple solutions can be used in the exact integration of the Lagrange subsystem (5.58).

There are two principle difficulties in the application of the finite group technique. The first is that of solving the sets of nonlinear partial differential equations which arise when the method is applied. As in the case of the infinitesimal method this difficulty is eased by the fact that possible transformations are always linear in the dependent variables and linear in one of the independent variables. Equation (4.83) is an example of such a transformation. The second difficulty is satisfaction of the closure property. For example, the determination of an appropriate transformation for the boundary layer equations leads to the restrictions given by the system of equations (4.55). For each choice of the function $h_1(x)$ in (4.55e), satisfaction of the closure property required extensive algebraic manipulations. In addition, choices of $h_1(x)$, such as the sin x, cause a very difficult problem when satisfying closure.

Because of the additional requirement of satisfying the closure property the finite group technique is more difficult to apply than the infinitesimal method. On the other hand, there are distinct differences in the results. When considering the Burgers' equation the invariant η of (4.29) is different from the invariants η_1 of (5.66) and η_3 of (5.79). For the boundary layer equations the similarity variable involving v can be contrasted by comparing (5.148) and (4.74) and the resulting ordinary differential equations (5.153) and (5.154) can be contrasted to (4.79) and (4.80).

These differences in difficulty of application and in the results suggest that there is some inherent dissimilarity between the finite group technique and the infinitesimal method. Perhaps some overlap in the results of the methods could be obtained by generalizing the second infinitesimal transformation in (5.1). This step is suggested by the fact that an infinitesimal transformation for the Burgers' equation can be found by converting the group (4.21) to infinitesimal form. The result is

$$\bar{x} = x + (x - y)\varepsilon$$
$$\bar{y} = y + 2y\,\varepsilon \qquad (6.1)$$
$$\bar{u} = u - (u + 1)\varepsilon.$$

As has already been noted the invariants (see (4.29)) corresponding to such a group are not found by the infinitesimal method of Chapter V.

From the above observations there are two conclusions. First, if an investigator is to obtain all possible results he must use both methods for obtaining similarity variables. Second, the infinitesimal method is simpler to apply and very often gives more results.

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13. ABSTRACT

The problem of developing systematic methods for obtaining similarity variables is considered for partial differential equations. Similarity variables are a set of transformations which reduce a partial differential equation to an ordinary differential equation.

This paper considers two methods of generating similarity variables. The first method uses a group of finite transformations and the second uses a group of infinitesimal transformations. The mathematical theory for both techniques is described and illustrated.

The two methods of obtaining similarity variables are applied to the Burgers' equation $u_y + uu_x = u_{xx}$ and to the laminar boundary layer equations with a pressure gradient. In all cases considered, new types of similarity variables are found. In addition, the auxiliary conditions are discussed in the light of the new similarity variables obtained for the boundary layer equations.

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