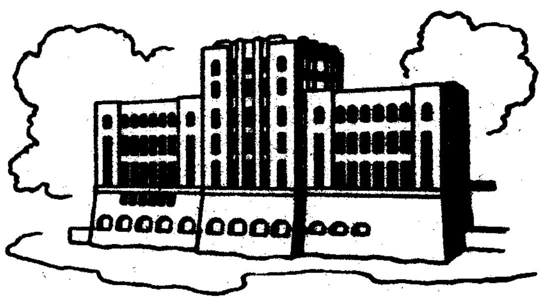
NATURAL FREQUENCIES OF A BODY OF REVOLUTION VIBRATING TRANSVERSELY IN A FLUID

by
Louis Landweber

This research was sponsored by the Department of Acoustics and Vibration of the Naval Ship Research and Development Center under Naval Ship Systems Command Subproject SF 113 1108, Task 1360, Contract Nonr-3271(01)(X).



IIHR Report No. 111

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1. Introduction

In a previous paper [1], a procedure for determining the natural frequencies of a body vibrating in a fluid was described and applied to a flexible circular cylinder. A more practical and more difficult application of the method, to the case of a body of revolution, is presented in the present work.

As was shown in [1], the natural frequencies are given by the eigenvalues of the potential energy matrix of the elastic body with respect to an inertia matrix, the latter being derived from the mass distribution of the body and the kinetic energy of the fluid. Thus two matrices must be obtained, and since the determination of the former is a problem in elasticity, and the determination of the latter one in hydrodynamics, these will be treated in separate sections. Then, in the final section, a particular body of revolution with prescribed elastic and inertial characteristics will be assumed, and its natural frequencies of vibration in air and in water will be calculated. For vibration in water, results obtained by means of strip theory, and by the present matrix technique will be compared.

2. Kinetic Energy of Fluid

Formulation of problem

Take the axis of symmetry of a body of revolution to extend from -1 to $+1$ along the x-axis of a cylindrical coordinate system (x, r, θ), and denote the equation of the body by

$$r^2 = y^2 + z^2 = f(x) \tag{1}$$

where

$$y = r \cos \theta, \quad z = r \sin \theta \tag{2}$$

Then, putting

$$\frac{dr}{dx} = \tan \gamma \tag{3}$$

we have for the direction cosines of the outward normal to the surface S of the body

$$\left.\begin{aligned} \frac{\partial x}{\partial n} &= -\frac{dr}{ds} = -\sin\gamma \\ \frac{\partial y}{\partial n} &= \cos\theta\cos\gamma \\ \frac{\partial z}{\partial n} &= \sin\theta\cos\gamma \end{aligned}\right\} \tag{4}$$

For a vertical vibration in the y-direction

$$\dot{y} = V(x)\ell^{i\omega t} \tag{5}$$

we may take the velocity potential (omitting the factor $\ell^{i\omega t}$) in the form

$$\Phi(x,r,\theta) = \phi(x,r)\cos\theta \tag{6}$$

where $\Phi(x,r,\theta)$ satisfies the Laplace equation and the boundary condition on the surface of the body

$$\frac{\partial\Phi}{\partial n} = V(x)\cos\theta\cos\gamma \tag{7}$$

Then we have

$$\frac{\partial\phi}{\partial n} = V(x)\cos\gamma \tag{8}$$

An integral equation for the velocity potential

Consider Green's reciprocal formula

$$\int_S \Phi \frac{\partial\Phi'}{\partial n}\,dS = \int_S \Phi' \frac{\partial\Phi}{\partial n}\,dS \tag{9}$$

where Φ and Φ' are potential functions which vanish sufficiently rapidly at infinity. If we select Φ' so that it is also of the form of (6),

$$\Phi'(x,r,\theta) = \Phi'(x,r)\cos\theta$$

then, by applying (8), we obtain from (9)

$$\int_{-1}^{1} r\phi \frac{\partial\phi'}{\partial n}\sec\gamma\,dx = \int_{-1}^{1} r\phi' V(x)dx \tag{10}$$

Let us take for Φ' the potential of a y-oriented doublet at the point $x = \xi$ on the axis of symmetry,

$$\Phi' = \frac{y}{R^3} = \frac{r \cos \theta}{R^3}, \qquad R = [(x - \xi)^2 + r^2(x)]^{1/2} \tag{11}$$

and hence

$$\phi'(x,r) = \frac{r}{R^3} \tag{12}$$

Then we have

$$\begin{aligned} \frac{\partial \phi'}{\partial r} &= \frac{\partial \phi'}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial n} = -\frac{\partial \phi'}{\partial x}\sin \gamma + \frac{\partial \phi'}{\partial n}\cos \gamma \\ &= \frac{3r(x - \xi)}{R^5}\sin \gamma + \left(\frac{1}{R^3} - \frac{3r^2}{R^5}\right)\cos \gamma \end{aligned} \tag{13}$$

and hence (10) becomes

$$\int_{-1}^{1} r\phi \, \frac{\frac{3}{2}(x - \xi)f'(x) + (x - \xi)^2 - 2r^2}{R^5}\, dx = \int_{-1}^{1} \frac{r^2}{R^3} V(x)dx \tag{14}$$

where

$$f'(x) = \frac{d}{dx} r^2(x)$$

The kernel

$$k(x,\xi) = \frac{1}{R^5}\left[\frac{3}{2} f'(x)(x - \xi) + (x - \xi)^2 - 2r^2\right] \tag{15}$$

can be written in the form

$$\left.\begin{aligned} k(x,\xi) &= \frac{1}{R^3}[3 \sin \alpha \cos \alpha \tan \gamma + 3 \cos^2\alpha - 2] \\ &= \frac{\sec \gamma}{R^3}[3 \cos \alpha \cos(\alpha - \gamma) - 2 \cos \gamma] \end{aligned}\right\} \tag{16}$$

where

$$x - \xi = R \cos \alpha, \qquad r(x) = R \sin \alpha \tag{17}$$

Except near the ends of the body, γ is small and we have, approximately,

$$k(x,\xi) \doteqdot \frac{1}{R^3}(3 \cos^2\alpha - 2) = \frac{\sin^3\alpha}{r^3(x)}(3 \cos^2\alpha - 2) \tag{18}$$

For values of x near ξ, $r(x)$ varies slowly (except near the ends of the body) as α varies from $\pi/4$ to $3\pi/4$. In the course of this variation, $k(x,\xi)$ assumes approximately the same initial and final values, but decreases to a minimum at $\alpha \doteq \pi/2$:

$$k(x,\xi) \doteq \begin{cases} -\dfrac{\sqrt{2}}{8r^3}, & \alpha = \dfrac{\pi}{4} \text{ or } \dfrac{3\pi}{4} \\ -\dfrac{2}{r^3}, & \alpha = \dfrac{\pi}{2} \end{cases} \tag{19}$$

Thus, if $r(x) \ll 1$, the graph of $k(x,\xi)$ against x would show a sharp trough in the neighborhood of $x = \xi$.

In order to examine the nature of the kernel near one end of the body, put

$$\xi = 1 - \xi', \qquad x = 1 - x', \qquad 0 < \xi', x' \ll 1$$

Also assume that in this neighborhood

$$r^2(x') = a_1 x' + a_2 x'^2 + \ldots \tag{20}$$

where a_1 and a_2 are of the same order of magnitude as ξ'. Then, noting that

$$r \tan \gamma = \frac{1}{2} f'(x)$$

and retaining terms to the second order, we can write (15) approximately in the form

$$k(x,\xi) \doteq \frac{\frac{3}{2}(x' - \xi')\, a_1 + (x' - \xi')^2 - 2a_1 x'}{[(x' - \xi')^2 + a_1 x']^{5/2}} \tag{21}$$

At $x' = 0, \xi'$, and $2\xi'$ we obtain the values

$$k(x,\xi) \doteq \begin{cases} \dfrac{\xi' - \frac{3}{2} a_1}{\xi'^4}, & x' = 0 \\ -\dfrac{2}{(a_1 \xi')^{3/2}}, & x' = \xi' \\ \dfrac{\xi' - \frac{5}{2} a_1}{\xi'^{3/2}(\xi' + 2a_1)^{5/2}}, & x' = 2\xi' \end{cases} \tag{21}$$

Each of these expressions is of the order of magnitude of $1/\xi'^3$.

The factor r^2/R^3 of the integrand in the right member of (14) can be analyzed in a similar way. We can write

$$\frac{r^2}{R^3} = \frac{\sin^3\alpha}{r(x)}$$

Then, except near the ends of the body, we can write

$$\frac{r^2}{R^3} = \begin{cases} \frac{\sqrt{2}}{4r}, & \alpha = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \\ \frac{1}{r}, & \alpha = \frac{\pi}{2} \end{cases} \tag{22}$$

and when x and ξ are near 1,

$$\frac{r^2}{R^3} \doteq \begin{cases} 0, & x' = 0 \\ \frac{1}{(a_1\xi')^{1/2}}, & x' = \xi \\ \frac{2a_1}{\sqrt{\xi'}\,(\xi' + 2a_1)^{3/2}}, & x' = 2\xi \end{cases} \tag{23}$$

Since a_1 is of the same order of magnitude as ξ', each of the expressions in (23) is of the order of magnitude of $1/\xi'$.

Numerical treatment of integrals

If the body of revolution is elongated $(r_{MAX} << 1)$, the foregoing analysis indicates that the integrands in (14) peak sharply in the neighborhood of $x = \xi$. Elsewhere the integrals in (14) can be readily replaced by a quadrature formula. This property suggests, however, that a good approximation to the value of the integral over a small interval about $x = \xi$ may be obtained as follows:

$$\int_{\xi_1}^{\xi_2} r(x)\phi(x)k(x,\xi)dx \doteq \int_{\xi_1}^{\xi_2} \left\{ r(\xi)\phi(\xi) + (x - \xi)\frac{d}{d\xi}[r(\xi)\phi(\xi)] \right\} k'(x,\xi)dx \tag{24}$$

where $k'(x,\xi)$ is an approximation to $k(x,\xi)$ given by

$$\left.\begin{aligned} k'(x,\xi) &= \frac{1}{R'^3} - \frac{1}{R'^5}\,[3f(\xi) + \tfrac{3}{2}(x - \xi)f'(\xi)] \\ R'^2 &= (x - \xi)^2 + f(\xi) + (x - \xi)f'(\xi) = (x - p)^2 + q \\ p &= \xi - \tfrac{1}{2}\,f'(\xi), \qquad q = f(\xi) - \tfrac{1}{4}\,f'(\xi)^2 \end{aligned}\right\} \tag{25}$$

Evaluating the integrals occurring in (24), we obtain

$$I_1 = \int_{\xi_1}^{\xi_2} k'(x,\xi)dx = \left(-\frac{1}{q}\frac{x - p}{R'} - \frac{x - \xi}{R'^3}\right)\Big|_{\xi_1}^{\xi_2} \tag{26}$$

and

$$I_2 = \int_{\xi_1}^{\xi_2} (x - \xi)k'(x,\xi)dx = -\frac{(x - \xi)^2}{R'^3}\Big|_{\xi_1}^{\xi_2} \tag{27}$$

in terms of which (24) may be written

$$\int_{\xi_1}^{\xi_2} r(x)\phi(x)k(x,\xi)dx \fallingdotseq r(\xi)\phi(\xi)I_1 + \frac{d}{d\xi}\,[r(\xi)\phi(\xi)]I_2 \tag{28}$$

Similarly, the integral on the right of (14), evaluated in a small interval about ξ, yields

$$\begin{aligned} \int_{\xi_1}^{\xi_2} \frac{r^2}{R^3}\,V(x)dx &\fallingdotseq \int_{\xi_1}^{\xi_2} \frac{1}{R'^3}\left\{f(\xi)V(\xi) + (x - \xi)\,\frac{d}{d\xi}\,[f(\xi)V(\xi)]\right\}dx \\ &\fallingdotseq f(\xi)V(\xi)J_1 + \frac{d}{d\xi}\,[f(\xi)V(\xi)]J_2 \end{aligned} \tag{29}$$

Here, by (25),

$$J_1 = \int_{\xi_1}^{\xi_2} \frac{dx}{[(x - p)^2 + q]^{3/2}} = \frac{x - p}{qR'}\Big|_{\xi_1}^{\xi_2} \tag{30}$$

$$J_2 = \int_{\xi_1}^{\xi_2} \frac{x - \xi}{[(x - p)^2 + q]^{3/2}}\,dx = -\frac{1}{R'}\Big|_{\xi_1}^{\xi_2} \tag{31}$$

Let us now replace the integrals in (14) by quadrature formulae. Subdivide the interval $-1 \le x \le 1$ into $n + 1$ segments of length $\Delta x = 2/(n + 1)$ at the points x_i, $i = 0, 1, 2, \ldots n + 1$, where $x_0 = -1$ and $x_{n+1} = 1$. Put $\xi = x_j$ and $\phi(x_i) = \phi_i$, $r(x_i) = r_i$, $k(x_i, x_j) = k_{ij}$, etc.

Using the trapezoidal rule, with $\xi_1 = x_{j-1}$ and $\xi_2 = x_{j+1}$ in (28), we obtain

$$\int_{-1}^{1} r\ \phi\ k(x,\xi)dx \doteq \Delta x[\sum_{i=1}^{n} r_i \phi_i k_{ij} - \frac{1}{2}(r_{j-1}\phi_{j-1}k_{j-1,j} + 2r_j\phi_j k_{jj} + r_{j+1}\phi_{j+1}k_{j+1,j}) + r_{j-1}\phi_{j-1}P_j + r_j\phi_j K_j - r_{j+1}\phi_{j+1}P_j] \tag{32}$$

where

$$K_j = -\frac{1}{q_j}\left[\frac{1}{R'_{1j}} + \frac{1}{R'_{2j}} - \frac{f_j'^2}{R'_{1j}R'_{2j}(R'_{1j} + R'_{2j})}\right] - \frac{1}{R'^3_{1j}} - \frac{1}{R'^3_{2j}} \tag{33}$$

$$P_j = \frac{1}{2}\left(\frac{1}{R'^3_{1j}} - \frac{1}{R'^3_{2j}}\right)$$

$$R'^2_{1j} = (\Delta x)^2 - f'_j\Delta x + f_j, \qquad R'^2_{2j} = (\Delta x)^2 + f'_j\Delta x + f_j \tag{34}$$

Similarly the right-hand member of (14) may be written

$$\int_{-1}^{1} \frac{r^2}{R^3} V(x)dx \doteq \Delta x[\sum_{i=1}^{n} \frac{f_i}{R_{ij}^3} V_i - \frac{1}{2}\left(\frac{f_{j-1}}{R_{j-1,j}^3} V_{j-1} + \frac{2f_j}{R_{jj}^3} V_j + \frac{f_{j+1}}{R_{j+1,j}^3} V_{j+1}\right) + f_{j-1}V_{j-1}Q_j + f_jV_jS_j - f_{j+1}V_{j+1}Q_j] \tag{35}$$

where

$$Q_j = -\frac{f'_j}{R'_{1j}R'_{2j}(R'_{1j} + R'_{2j})\Delta x} \tag{36}$$

$$S_j = \frac{1}{q_j}\left[\frac{1}{R'_{1j}} + \frac{1}{R'_{2j}} - \frac{f_j'^2}{R'_{1j}R'_{2j}(R'_{1j} + R'_{2j})}\right] \tag{37}$$

Thus (14) may be expressed in the form

$$\sum_{i=1}^{n} r_i \phi_i H_{ij} = \sum_{i=1}^{n} V_i W_{ij} \tag{38}$$

where

$$H_{ij} = \begin{cases} k_{ij} & , \quad i \neq j-1, j, j+1 \\ \frac{1}{2} k_{j-1,j} + P_j & , \quad i = j-1 \\ K_j & , \quad i = j \\ \frac{1}{2} k_{j+1,j} - P_j & , \quad i = j+1 \end{cases} \tag{39}$$

and

$$W_{ij} = \begin{cases} \frac{f_i}{R_{ij}^3} & , \quad i \neq j-1, j, j+1 \\ \frac{1}{2} \frac{f_{j-1}}{R_{j-1,j}^3} + Q_j & , \quad i = j-1 \\ S_j & , \quad i = j \\ \frac{1}{2} \frac{f_{j+1}}{R_{j+1,j}^3} - Q_j & , \quad i = j+1 \end{cases} \tag{40}$$

Added-mass matrix

The maximum value of the kinetic energy of the fluid during a vibration cycle is given by

$$2T_F = -\rho \int_S \Phi \frac{\partial \Phi}{\partial n} dS \tag{41}$$

or, by (6) and (7),

$$2T_F = -\pi\rho \int_{-1}^{1} r \phi V(x) dx \tag{42}$$

where ρ is the mass density of the fluid. By applying the trapezoidal rule, (42) can be expressed in the form

$$2T_F = -\pi\rho\Delta x \sum_{j=1}^{n} r_j \phi_j V_j \tag{43}$$

We wish to eliminate the quantities $r_j \phi_j$ in (43). For this purpose we multiply (38) by the inverse matrix H_{jk}^{-1} of H_{ij} such that

$$\sum_{j=1}^{n} H_{ij} H_{jk}^{-1} = \delta_{ik}$$

where δ_{ik} is the Kronecker delta. This gives

$$r_j \phi_j = \sum_i \sum_k W_{ik} H_{kj}^{-1} V_i \tag{44}$$

which, substituted into (43), yields the quadratic form

$$\left.\begin{aligned} 2T_F &= \Delta x \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} V_i V_j \\ A_{ij} &= -\frac{\pi\rho}{2} \sum_{k=1}^{n} [W_{ik} H_{kj}^{-1} + W_{jk} H_{ki}^{-1}] \end{aligned}\right\} \tag{45}$$

where A_{ij} is the desired added-mass matrix.

Note that A_{ij} has the dimensions of mass per unit length. For a body of length L, the corresponding added-mass matrix is given by

$$A'_{ij} = \frac{L^2}{4} A_{ij} \tag{45a}$$

It will be of interest later to compare the result in (45) with that from "strip theory," in which an added-mass coefficient of unity is assumed at each circular section,

$$A_{ij}^{(s)} = \pi\rho f_i \delta_{ij} \tag{45b}$$

3. Potential Energy of Elastic Body

Potential energy of bending

The potential energy of bending of a body of length L is [2]

$$V_B = \int_0^L \frac{EI}{2} y''^2 dx \tag{46}$$

where E is the modulus of elasticity, I is the moment of inertia of the section area about a diameter, and $y'' = d^2y/dx^2$. The free-free end conditions are

$$y''(0) = y'''(0) = y''(L) = y'''(L) = 0 \tag{47}$$

Let us divide the interval $0 \le x \le L$ into $n + 1$ sections of length

$$h = \frac{L}{n + 1} \tag{48}$$

at the points x_i, $i = 0, 1, 2, \ldots n + 1$, and put $y_i = y(x_i)$, $y''(x_i) = y''_i$, $EI(x_i) = G_i$. The free-free conditions will be satisfied approximately by putting

$$y''_0 = y''_1 = y''_n = y''_{n+1} = 0 \tag{49}$$

Setting

$$y''_i = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1}) \tag{50}$$

and applying the trapezoidal rule, (46) may be written as

$$V_B = \frac{1}{2h^3} \sum_{i=2}^{n-1} (y_{i+1} - 2y_i + y_{i-1})^2 G_i \tag{51}$$

The $n + 2$ variables y_0, y_1, $\ldots y_n$, y_{n+1} are not all independent since they are subject to the four conditions (49). We may use these conditions to express y_0, y_1, y_n, y_{n+1} in terms of y_2, y_3, $\ldots y_{n-1}$. Since y_0 and y_{n+1} do not appear in (51), we need consider only the elimination of y_1 and y_n.

Applying the condition $y''_1 = 0$, we may write

$$y \doteq y_1 + a(x - x_1) + b(x - x_1)^3 \tag{52}$$

for the neighborhood of y_1. Then we have

$$y_2 = y_1 + ah + bh^3$$

$$y_3 = y_1 + 2ah + 8bh^3$$

$$y_4 = y_1 + 3ah + 27bh^3$$

Hence, eliminating a and b, we obtain,

$$y_1 = \begin{vmatrix} y_2 & 1 & 1 \\ y_3 & 2 & 8 \\ y_4 & 3 & 27 \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 8 \\ 1 & 3 & 27 \end{vmatrix}$$

or

$$y_1 = \frac{1}{2}(5y_2 - 4y_3 + y_4) \tag{53}$$

Similarly

$$y_n = \frac{1}{2}(5y_{n-1} - 4y_{n-2} + y_{n-3}) \tag{54}$$

We now have, by (53) and (54),

$$y_1 - 2y_2 + y_3 = \frac{1}{2}(y_2 - 2y_3 + y_4)$$

$$y_n - 2y_{n-1} + y_{n-2} = \frac{1}{2}(y_{n-1} - 2y_{n-2} + y_{n-3})$$

and hence (51) becomes

$$2V_B = \frac{1}{h^3}\sum_{i=3}^{n-2}(y_{i+1} - 2y_i + y_{i-1})^2 G_i' = \frac{1}{h^3}\sum_{i,j=2}^{n-1} h_{ij} y_i y_j \tag{55}$$

where

$G_3' = G_3 + \frac{1}{4} G_2$, G_{n-2}', $G_{n-2}' = G_{n-2} + \frac{1}{4} G_{n-1}$; $G_i' = G_i$, $i = 4, 5, \ldots n-3$

and $h_{ij} = h_{ji}$ and is nonzero only for

$$\begin{aligned} h_{ii} &= G_{i+1}' + 4G_i' + G_{i-1}', & i &= 2, 3, \ldots n-1 \\ h_{i,i+1} &= -2(G_i' + G_{i+1}'), & i &= 2, 3, \ldots n-2 \\ h_{i,i+2} &= G_{i+1}', & i &= 2, 3, \ldots n-3 \end{aligned} \tag{56}$$

In applying (56) we must set

$$G_1' = G_2' = G_{n-1}' = G_n' = 0 \tag{57}$$

since only G_3', G_4', . . . G_{n-2}' occur in (55).

An interesting property of the matrix (h_{ij}) is that it is singular, and that its rank does not exceed $n - 4$.

In order to prove this property, consider the determinant $|h_{ij}|$, $i,j = 2,3, \ldots n-1$. We may replace the elements h_{2j} of the first row by the sum of the elements over all the rows. By (56) and (57) this yields

$$\sum_{i=2}^{n-1} h_{ij} = G'_{j-1} - 2(G'_{j-1} + G'_j) + (G'_{j+1} + 4G'_j + G'_{j-1}) - 2(G'_j + G'_{j+1}) + G'_{j+1} = 0$$

i.e. the elements of the first row are now all zeros. Next multiply the elements of the i-th row by $i - 1$ and replace the elements of the last row by the sum of the resulting elements in all the rows, to obtain

$$\sum_{i=2}^{n-1} (i-2)h_{ij} = (j-4)G'_{j-1} - 2(j-3)(G'_{j-1} + G'_j) + (j-2)(G'_{j+1} + 4G'_j + G'_{j-1}$$

$$- 2(j-1)(G'_j + G'_{j+1}) + jG'_{j+1} = 0$$

Thus the elements of the last row have also been replaced by zeros, showing that the rank of the matrix (h_{ij}) is at most $n - 4$, as was stated.

We shall also need to eliminate V_1 and V_n from the quadratic form for the kinetic energy

$$2T = h \sum_{i,j=1}^{n} a_{ij} V_i V_j = h \sum_{i,j=2}^{n-1} b_{ij} V_i V_j \tag{58}$$

by means of the relations corresponding to (53) and (54),

$$\left.\begin{aligned} V_1 &= \frac{1}{2}(5V_2 - 4V_3 + V_4) \\ V_n &= \frac{1}{2}(5V_{n-1} - 4V_{n-2} + V_{n-3}) \end{aligned}\right\} \tag{59}$$

We obtain: $\qquad b_{ij} = a_{ij}\,, \quad 5 \le i,j \le n-4$

$$b_{22} = a_{22} + \frac{25}{4} a_{11} + 5a_{12} \qquad b_{n-1,n-1} = a_{n-1,n-1} + \frac{25}{4} a_{nn} + 5a_{n,n-1}$$

$$b_{23} = a_{23} - 5a_{11} - 2a_{12} + \frac{5}{2} a_{13} \qquad b_{n-1,n-2} = a_{n-1,n-2} - 5a_{nn} - 2a_{n,n-1} + \frac{5}{2} a_{n,n-2}$$

$$b_{24} = a_{24} + \frac{5}{4} a_{11} + \frac{1}{2} a_{12} + \frac{5}{2} a_{14} \qquad b_{n-1,n-3} = a_{n-1,n-3} + \frac{5}{4} a_{nn} + \frac{1}{2} a_{n,n-1} + \frac{5}{2} a_{n,n-3}$$

$$b_{33} = a_{33} + 4a_{11} - 4a_{13} \qquad b_{n-2,n-2} = a_{n-2,n-2} + 4a_{nn} - a_{n,n-2}$$

$$b_{34} = a_{34} - a_{11} + \frac{1}{2} a_{13} - 2a_{14} \qquad b_{n-2,n-3} = a_{n-2,n-3} - a_{nn} + \frac{1}{2} a_{n,n-2} - 2a_{n,n-3}$$

$$b_{44} = a_{44} + \frac{1}{4} a_{11} + a_{14} \qquad b_{n-3,n-3} = a_{n-3,n-3} + \frac{1}{4} a_{nn} + a_{n,n-3}$$

$$\left.\begin{array}{ll} b_{2j} = a_{2j} + \frac{5}{2} a_{ij} , & b_{n-1,j} = a_{n-1,j} + \frac{5}{2} a_{nj} \\ b_{3j} = a_{3j} - 2a_{ij} , & b_{n-2,j} = a_{n-2,j} - 2a_{nj} \\ b_{4j} = a_{4j} + \frac{1}{2} a_{ij} , & b_{n-3,j} = a_{n-3,j} + \frac{1}{2} a_{nj} \end{array}\right\} \quad 5 \le j \le n-4$$

$$b_{2,n-1} = a_{2,n-1} + \frac{5}{2} a_{1,n-1} + \frac{25}{4} a_{1n} + \frac{5}{2} a_{2n}$$

$$b_{2,n-2} = a_{2,n-2} + \frac{5}{2} a_{1,n-2} - 5a_{1n} - 2a_{2n}$$

$$b_{2,n-3} = a_{2,n-3} + \frac{5}{2} a_{1,n-3} + \frac{5}{4} a_{1n} + \frac{1}{2} a_{2n}$$

$$b_{3,n-1} = a_{3,n-1} - 2a_{1,n-1} - 5a_{1n} + \frac{5}{2} a_{3n}$$

$$b_{3,n-2} = a_{3,n-2} - 2a_{1,n-2} + 4a_{1n} - 2a_{3n}$$

$$b_{3,n-3} = a_{3,n-3} - 2a_{1,n-3} - a_{1n} + \frac{1}{2} a_{3n}$$

$$b_{4,n-1} = a_{4,n-1} + \frac{1}{2} a_{1,n-1} + \frac{5}{4} a_{1n} + \frac{5}{2} a_{4n}$$

$$b_{4,n-2} = a_{4,n-2} + \frac{1}{2} a_{1,n-2} - a_{1n} - 2a_{4n}$$

$$b_{4,n-3} = a_{4,n-3} + \frac{1}{2} a_{1,n-3} + \frac{1}{4} a_{1n} + \frac{1}{2} a_{4n} \qquad (60)$$

If a_{ij} is a diagonal matrix, (60) becomes

$$\left.\begin{aligned}
b_{22} &= a_{22} + \tfrac{25}{4} a_{11} & b_{n-1,n-1} &= a_{n-1,n-1} + \tfrac{25}{4} a_{nn} \\
b_{23} &= -5a_{11} & b_{n-1,n-2} &= -5a_{nn} \\
b_{24} &= \tfrac{5}{4} a_{11} & b_{n-1,n-3} &= \tfrac{5}{4} a_{nn} \\
b_{33} &= a_{33} + 4a_{11} & b_{n-2,n-2} &= a_{n-2,n-2} + 4a_{nn} \\
b_{34} &= -a_{11} & b_{n-2,n-3} &= -a_{nn} \\
b_{44} &= a_{44} + \tfrac{1}{4} a_{11} & b_{n-3,n-3} &= a_{n-3,n-3} + \tfrac{1}{4} a_{nn} \\
& & b_{ij} = a_{ii}\delta_{ij}\ , \quad & 5 \le i,j \le n-4
\end{aligned}\right\} \tag{60a}$$

Potential energy of bending and shear

The shearing force F is related to the bending moment M by the relation

$$F = -\frac{dM}{dx} \tag{61}$$

Let y_B denote the deflection due to bending, y_s that due to shear, so that the total deflection y is given by

$$y = y_B + y_s \tag{62}$$

Here y_B satisfies the free-free conditions (49). Then we have

$$M = EIy_B'' , \qquad F = -EIy_B''' \tag{63}$$

The shearing force can also be expressed in the form

$$F = k\ A\ G\ \beta\ , \qquad \beta = y_s' \tag{64}$$

where A is the area of transverse section, G is the shear modulus and k is a numerical factor, given by $k = \frac{1}{2}$ for a circular section. Equating the expressions for F in (63) and (64), and applying (62), we obtain the relation

$$-EIy_B''' + k\ A\ G\ y_B' = k\ A\ G\ y' \tag{65}$$

The potential energy of bending, given in (46), must now be replaced by

$$V_B = \frac{1}{2}\int_0^L EIy_B''^2 dx \tag{66}$$

The potential energy of shear is given by

$$V_s = \int_0^L\int_0^\beta Fd\beta dx = \frac{1}{2}\int_0^L kAG\beta^2 dx$$

or, by (63) and (64),

$$V_s = \frac{1}{2}\int_0^L \frac{(EI)^2}{kAG} y_B'''^2 dx \tag{67}$$

Hence the total potential energy is

$$V = V_B + V_s = \frac{1}{2}\int_0^L [EIy_B''^2 + \frac{(EI)^2}{kAG} y_B'''^2]dx \tag{68}$$

The discretization procedure for V_B yields the same form as in (55)

$$2V_B = \frac{1}{h^3}\sum_{i,j=2}^{n-1} h_{ij}y_{Bi}y_{Bj} \tag{69}$$

For V_s we have by the trapezoidal rule

$$2V_s = h\sum_{i=1}^{n} H_i y_{Bi}'''^2 \ , \qquad H_i = \frac{G_i^2}{kGA_i} \tag{70}$$

Here $i = 0$ and $n + 1$ are excluded because of the free-free condition (47). Since $y_B'''(0) = 0$, we may assume, for the neighborhood of $x = 0$, that $y_B''(x)$ is of the form

$$y_B''(x) = cx^2$$

and then

$$y_B'''(x) = 2cx$$

i.e. $y_B'''(x)$ is proportional to x near $x = 0$. Hence we have

$$y_{B1}''' \doteq \frac{1}{2} y_{B2}''' \tag{71}$$

At $i = 2.5$ we may put

$$y_{B2\frac{1}{2}}''' = \frac{1}{h^3} (y_4 - 3y_3 + 3y_2 - y_1)_B$$

We shall assume then

$$y_{B1}''' = \frac{0.4}{h^3} (y_4 - 3y_3 + 3y_2 - y_1)_B$$

or, by (53),

$$y_{B1}''' = \frac{0.2}{h^3} (y_2 - 2y_3 + y_4)_B \tag{72}$$

and, by (71),

$$y_{B2}''' = \frac{0.4}{h^3} (y_2 - 2y_3 + y_4)_B \tag{73}$$

Similarly,

$$y_{Bn}''' = \frac{0.2}{h^3} (y_n - 2y_{n-1} + y_{n-2})_B \tag{74}$$

$$y_{B,n-1}''' = \frac{0.4}{h^3} (y_n - 2y_{n-1} + y_{n-2})_B \tag{75}$$

For y_{Bi}''' , $i = 3, 4, \ldots n - 2$, we have the central-difference formula

$$y_{Bi}''' = \frac{1}{2h^3} (y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})_B \tag{76}$$

Since y_{1B} and y_{nB} occur in these expressions for y_{B3}''' and $y_{B,n-2}'''$, we again apply (53) and (54), which yield

$$y_{B3}''' = \frac{1}{4h^3} (2y_5 - 5y_4 + 4y_3 - y_2)_B \tag{77}$$

$$y_{B,n-2}''' = \frac{1}{4h^3} (2y_{n-4} - 5y_{n-3} + 4y_{n-2} - y_{n-1})_B \tag{78}$$

Applying (76), we obtain

$$h \sum_{i=4}^{n-3} H_i y_{Bi}'''^2 = \frac{1}{4h^5} \sum_{i,j=2}^{n-1} c_{ij} y_{Bi} y_{Bj} \tag{79}$$

where $c_{ij} = c_{ji}$, and

$$\left.\begin{aligned}
c_{ii} &= H_{i-2} + 4H_{i-1} + 4H_{i+1} + H_{i+2}\,, && i = 2,3,\ldots n-1\\
c_{i,i+1} &= -2(H_{i-1} + H_{i+2})\,, && i = 2,3,\ldots n-2\\
c_{i,i+2} &= -4H_{i+1}\,, && i = 2,3,\ldots n-3\\
c_{i,i+3} &= 2(H_{i+1} + H_{i+2})\,, && i = 2,3,\ldots n-4\\
c_{i,i+4} &= -H_{i+2}\,, && i = 2,3,\ldots n-5
\end{aligned}\right\} \qquad (80)$$

Since only H_4, H_5, . . . H_{n-3} occur in (79), we must set in (80)

$$H_0 = H_1 = H_2 = H_3 = H_{n-2} = H_{n-1} = H_n = H_{n+1} = 0 \qquad (81)$$

To (79) we must now add

$$h[H_1 y'''^{\,2}_{B1} + H_2 y'''^{\,2}_{B2} + H_3 y'''^{\,2}_{B3} + H_{n-2} y'''^{\,2}_{B,n-2} + H_{n-1} y'''^{\,2}_{B,n-1} + H_n y'''^{\,2}_{B,n}]$$

From (72), (73), and (77) we have (with H_1, H_2, $H_3 \neq 0$)

$$h(H_1 y'''^{\,2}_{B1} + H_2 y'''^{\,2}_{B2} + H_3 y'''^{\,2}_{B3})$$

$$= \frac{1}{h^5}\,[(.04H_1 + .16H_2)(y_2^{\,2} + 4y_3^{\,2} + y_4^{\,2} - 4y_2y_3 + 2y_2y_4 - 4y_3y_4)$$

$$+ \frac{H_3}{16}\,(4y_5^{\,2} + 25y_4^{\,2} + 16y_3^{\,2} + y_2^{\,2} - 8y_2y_3 + 10y_2y_4 - 4y_2y_5 - 40y_3y_4 + 16y_3y_5 - 20y_4y_5)]_B$$

Hence, putting

$$2V_s = \frac{1}{4h^5}\sum_{i,j=2}^{n-1}(c_{ij} + d_{ij})y_{Bi}y_{Bj} \qquad (82)$$

we have, with $H_2' = 0.16\,H_1 + 0.64\,H_2$,

$$\left.\begin{aligned}
c_{22} + d_{22} &= H_2' + \frac{H_3}{4} + H_4 \\
c_{23} + d_{23} &= -2H_2' - H_3 - 2H_4 \\
c_{24} + d_{24} &= H_2' + \frac{5}{4} H_3 \\
c_{25} + d_{25} &= -\frac{H_3}{2} + 2H_4 \\
c_{33} + d_{33} &= 4H_2' + 4H_3 + 4H_4 + H_5 \\
c_{34} + d_{34} &= -2H_2' - 5H_3 - 2H_5 \\
c_{35} + d_{35} &= 2H_3 - 4H_4 \\
c_{44} + d_{44} &= H_2' + \frac{25}{4} H_3 + 4H_5 + H_6 \\
c_{45} + d_{45} &= -\frac{5}{2} H_3 - 2H_6 \\
c_{55} + d_{55} &= H_3 + 4H_4 + 4H_6 + H_7
\end{aligned}\right\} \tag{83}$$

and similarly, with $H'_{n-1} = 0.16H_n + 0.64H_{n-1}$,

$$\left.\begin{aligned}
c_{n-1,n-1} + d_{n-1,n-1} &= H'_{n-1} + \frac{H_{n-2}}{4} + H_{n-3} \\
c_{n-1,n-2} + d_{n-1,n-2} &= -2H'_{n-1} - H_{n-2} - 2H_{n-3} \\
c_{n-1,n-3} + d_{n-1,n-3} &= H'_{n-1} + \frac{5}{4} H_{n-2} \\
c_{n-1,n-4} + d_{n-1,n-4} &= -\frac{1}{2} H_{n-2} + 2H_{n-3} \\
c_{n-2,n-2} + d_{n-2,n-2} &= 4H'_{n-1} + 4H_{n-2} + 4H_{n-3} + H_{n-4}
\end{aligned}\right\} \tag{84}$$

$$\left.\begin{aligned}
c_{n-2,n-3} + d_{n-2,n-3} &= -2H'_{n-1} - 5H_{n-2} - 2H_{n-4} \\
c_{n-2,n-4} + d_{n-2,n-4} &= 2H_{n-2} - 4H_{n-3} \\
c_{n-3,n-3} + d_{n-3,n-3} &= H'_{n-1} + \frac{25}{4} H_{n-2} + 4H_{n-4} + H_{n-5} \\
c_{n-3,n-4} + d_{n-3,n-4} &= -\frac{5}{2} H_{n-2} - 2H_{n-5} \\
c_{n-4,n-4} + d_{n-4,n-4} &= H_{n-2} + 4H_{n-3} + 4H_{n-5} + H_{n-6}
\end{aligned}\right\} \qquad (84)$$

Furthermore

$$d_{ij} = 0 , \qquad 5 < i \text{ or } j < n - 4 \qquad (85)$$

i.e. the elements of the potential energy matrix in (82), other than those given in (83) and (84), can be calculated from (80).

We shall now show that the matrix $(c_{ij} + d_{ij})$ has the same property as (h_{ij}); i.e., its rank does not exceed $n - 4$. Applying (80), (83), (84) and (85), we see that the sum of the elements of the j-th column of $(c_{ij} + d_{ij})$ is zero. This is certainly the case for a column in which (72) is satisfied, since

$$\sum_{i=2}^{n-1} c_{ij} = -H_{j-2} + 2(H_{j-2} + H_{j-1}) - 4H_{j-1} - 2(H_{j-2} + H_{j+1}) + H_{j-2} + 4H_{j-1}$$

$$+ 4H_{j+1} + 4H_{j+2}) - 2(H_{j-1} + H_{j+2}) - 4H_{j+1} + 2(H_{j+1} + H_{j+2}) - H_{j+2} = 0$$

For each column containing nonzero elements, d_{ij}, one can verify that the sum of the elements also vanishes; e.g.

$$(c_{23} + d_{23}) + (c_{33} + d_{33}) + (c_{43} + d_{43}) + (c_{53} + d_{53}) + c_{63} + c_{73}$$

$$= (-2H'_2 - H_3 - 2H_4) + (4H'_2 + 4H_3 + 4H_4 + H_5) + (-2H'_2 - 5H_3 - 2H_5)$$

$$+ (2H_3 - 4H_4) + 2(H_4 + H_5) - H_5 = 0$$

Thus the elements of the first row of the determinant $|c_{ij} + d_{ij}|$ may be replaced by zeros. Next, multiplying the elements of the i-th row by $i - 1$, and replacing the elements of the last row by the sum of the resulting elements in each column, we find that each sum is zero. This is readily verified for a column in which (72) is satisfied, since

$$\sum_{i=2}^{n-1}(i-2)c_{ij} = -(j-6)H_{j-2} + 2(j-5)(H_{j-2}+H_{j-1}) - 4(j-4)H_{j-1}$$

$$-2(j-3)(H_{j-2}+H_{j+1}) + (j-2)(H_{j-2}+4H_{j-1}+4H_{j+1}+H_{j+2}) - 2(j-1)(H_{j-1}+H_{j+2})$$

$$-4jH_{j+1} + 2(j+1)(H_{j+1}+H_{j+2}) - (j+2)H_{j+2} = 0$$

and also shown by inspection to be valid for columns containing nonzero elements d_{ij}; e.g., for the case considered above,

$$(4H_2' + 4H_3 + 4H_4 + H_5) + 2(-2H_2' - 5H_3 - 2H_5) + 3(2H_3 - 4H_4) + 8(H_4 + H_5) - 5H_5 = 0$$

Thus the elements of the last row of the determinant $|c_{ij} + d_{ij}|$ may also be replaced by zeros. This completes the proof that the rank of the matrix $(c_{ij} + d_{ij})$ does not exceed $n - 4$.

Kinetic energy corresponding to bending and shear

Expression (58) for the kinetic energy

$$2T = \sum_{i,j=1}^{n} a_{ij}V_iV_j$$

is a quadratic form in terms of the deflection velocities, and that for the potential energy in (82), is in terms of the deflections due to bending alone. In order to apply the present method for determining natural frequencies of vibration, we must express both the kinetic and potential energies in terms of the same variables; i.e. we must express either the y_{Bi} in (82) in terms of the y_i, or the V_i in (58) in terms of V_{Bi}. Let us select the latter alternative.

It will be convenient to write (65) in the approximate form

$$y' \doteq y_B' - \frac{EI}{kAG}\,y_B''' - \frac{d}{dx}\left(\frac{EI}{kAG}\right)y_B''$$

in which the last, additional term is small near the body ends because of the free-free condition (47), and small over the remainder of the body, where EI/kAG is nearly constant. Integration of this equation now yields

$$y = y_B - \frac{EI}{kAG} y''_B \tag{86}$$

in which the constant of integration is set equal to zero on the assumption that the end deflections of a free-free beam are unaffected by shear. In discrete form (86) becomes

$$y_i = y_{Bi} - \lambda_i(y_{B,i-1} - 2y_{Bi} + y_{B,i+1}), \qquad \lambda_i = \frac{EI_i}{kGA_i}, \qquad i = 1, 2, \ldots n \tag{87}$$

The free-free conditions (49) will be satisfied in (87) if we put

$$\lambda_0 = \lambda_1 = \lambda_n = \lambda_{n+1} = 0 \tag{88}$$

Differentiating (87) with respect to time now yields

$$V_i = -\lambda_i V_{B,i-1} + (1 + 2\lambda_i)V_{Bi} - \lambda_i V_{B,i+1} \tag{89}$$

Then we have

$$\begin{aligned} V_iV_j = [&\lambda_i\lambda_j V_{i-1}V_{j-1} + (1 + 2\lambda_i)(1 + 2\lambda_j)V_iV_j + \lambda_i\lambda_j V_{i+1}V_{j+1} - \lambda_i(1 + 2\lambda_j)V_{i-1}V_j \\ &- \lambda_j(1 + 2\lambda_i)V_iV_{j-1} - \lambda_i(1 + 2\lambda_j)V_{i+1}V_j - \lambda_j(1 + 2\lambda_i)V_iV_{j+1} \\ &+ \lambda_i\lambda_j V_{i-1}V_{j+1} + \lambda_i\lambda_j V_{i+1}V_{j-1}]_B \end{aligned}$$

Hence, writing

$$2T = \sum_{i,j=1}^{n} a_{ij}V_iV_j = \sum_{i,j=1}^{n} a'_{ij}V_{Bi}V_{Bj} \tag{90}$$

we obtain

$$\begin{aligned} a'_{ij} = &\lambda_{i+1}\lambda_{j+1}a_{i+1,j+1} + (1 + 2\lambda_i)(1 + 2\lambda_j)a_{ij} + \lambda_{i-1}\lambda_{j-1}a_{i-1,j-1} \\ &- \lambda_{i+1}(1 + 2\lambda_j)a_{i+1,j} - \lambda_{j+1}(1 + 2\lambda_i)a_{i,j+1} - \lambda_{i-1}(1 + 2\lambda_j)a_{i-1,j} \\ &- \lambda_{j-1}(1 + 2\lambda_i)a_{i,j-1} + \lambda_{i+1}\lambda_{j-1}a_{i+1,j-1} + \lambda_{i-1}\lambda_{j+1}a_{i-1,j+1} \end{aligned} \tag{91}$$

in which (88) is applicable.

Finally, after the a'_{ij} have been computed, the kinetic energy can be expressed in the form

$$2T = \sum_{i,j=2}^{n-1} b'_{ij}V_{Bi}V_{Bj} \tag{92}$$

in which the b'_{ij} are given in terms of the a'_{ij} by means of formulae (60).

4. Characteristic Equations

For the total kinetic energy of the body T_B we can write

$$2T_b = \int_0^L m(x)V^2dx$$

or in discrete form, by the trapezoidal rule,

$$2T_b = h\sum_{i=1}^{n} m_i V_i{}^2 \tag{93}$$

Hence, by (45a), we have for the total kinetic energy T

$$2T = h\sum_{i,j=1}^{n} [m_i\delta_{ij} + A'_{ij}]V_iV_j \tag{94}$$

(a) Vibration in air, neglecting shear

The potential-energy matrix, given in (55), is $\frac{1}{h^3}(h_{ij})$. Since the kinetic-energy matrix for this case is $h(m_i\delta_{ij})$, application of the transformation (60a), with m_i substituted for a_{ii}, yields the corresponding kinetic energy matrix, hb_{ij}. Application of the matrix method described in [1] now gives the characteristic determinant equation

$$|\omega^2hb_{ij} - \frac{1}{h^3}h_{ij}| = 0$$

$$|p^2b_{ij} - h_{ij}| = 0 \tag{95}$$

where, with f the frequency of vibration,

$$f = \frac{p}{2\pi h^2}, \qquad p = h^2\omega \tag{96}$$

By employing the procedure used to prove that the rank of (h_{ij}) does not exceed $n - 4$, the determinant in (95) is transformed into one with p^2 a factor of the elements of the first and last rows. Hence two of the eigenvalues p^2 of (h_{ij}) with respect to (b_{ij}) are zero, and there are at most $n - 4$ nonzero eigenvalues.

(b) Vibration in water (strip theory), neglecting shear

As in the previous case, the potential-energy matrix, $\frac{1}{h^3}(h_{ij})$, is given in (55). The added mass of a segment of radius r and thickness h is, according to strip theory, given by $\pi\rho hr^2$, where ρ is the density of the fluid. Hence the kinetic-energy matrix is now

$$(h(m_i + \pi\rho r_i^2)\delta_{ij})$$

Substituting $m_i + \pi\rho r_i^2$ for a_{ii} in (60a) yields the corresponding matrix b_{ij}, and formally the same characteristic equation (95). In this case also there are at most $n - 4$ nonzero eigenvalues.

(c) Vibration in water (added-mass matrix), neglecting shear

We now apply the kinetic-energy matrix given in (94). Substituting $a_{ij} = m_i\delta_{ij} + A'_{ij}$ into (60) yields the matrix b_{ij} and again, formally, the characteristic equation (95), which will have at most $n - 4$ nonzero eigenvalues.

(d) Vibration in air, with shear

By (69) and (82) we have for the potential-energy matrix

$$\frac{1}{h^3} h_{ij} + \frac{1}{4h^5}(c_{ij} + d_{ij})$$

From the kinetic-energy matrix $(a_{ij}) = m_i\delta_{ij}$ we first obtain (a'_{ij}) from (91), and then transform the (a'_{ij}) to (b'_{ij}) by (60). The resulting characteristic equation,

$$|p^2 b'_{ij} - (h_{ij} + \frac{c_{ij} + d_{ij}}{4h^2})| = 0 \tag{97}$$

again has at most $n - 4$ eigenvalues, as can be shown by the same procedure as was applied to (95).

(e) Vibration in water (strip theory), with shear

The potential-energy matrix is identical to that in the previous case, and the kinetic-energy matrix is the same as in (b). By successive transformations of $a_{ij} = (m_i + \pi\rho r_i^2)\delta_{ij}$ through (91) and (60) we obtain

(b'_{ij}) which is then substituted into the characteristic equation (97). Again this has at most $n - 4$ nonzero eigenvalues.

(f) Vibration in water (added-mass matrix), with shear

The potential-energy matrix is the same as in cases (d) and (e), and the kinetic-energy matrix is $(a_{ij}) = (m_i \delta_{ij} + A'_{ij})$. Transforming this into (b'_{ij}) by (91) and (60) gives the final characteristic equation (96), which also has at most $n - 4$ nonzero eigenvalues.

5. Computer Programs

Computer programs for calculating the natural frequencies of a given body are given in Appendix 1. All programs are in double-precision arithmetic.

Program 1 yields the added-mass matrix a_{ij} of equation (45a) in mass units based on the long ton (2240 lb) as the unit of weight. Note that i,j vary from 1 to 19 in Program 1, corresponding to the division of the body into 20 sections. The constant FO between items 11 and 12 is the quantity

$$FO = \frac{1.94}{2240} \frac{L^3}{8}$$

where L is the length of the body, taken to be 66.37 ft.

Program 2 yields the mass matrices b_{ij} of (95) for the three cases (a) vibration in air, (b) vibration in water, strip theory, (c) vibration in water, added-mass matrix, all with shear neglected. The corresponding mass matrices b'_{ij} when the effects of shear are included are given in Program 3.

Program 4 gives both the potential-energy matrix h_{ij} of (55) and (95) when shear is neglected and that for shear, $(c_{ij} + d_{ij})/4h^2$, of (97).

Finally the natural frequencies for the 6 cases are obtained from Program 5, in which the eigenvalues of the characteristic equations (95) and (97) are derived by calling for an available subroutine in IBM Manual "System/ 360 Scientific Subroutine Package, Number H20-0205-3.

6. Numerical Example

The foregoing procedures were applied to an assumed body.

The added-mass matrix a_{ij} of (45a) is given in Table 1. Kinetic-energy matrices b_{ij} of (58) and (95) for the three cases of vibration in air, in water by strip theory, and in water, employing the added-mass matrix, all with shear neglected, are given in Tables 2a, 2b and 2c, respectively. The corresponding kinetic-energy matrices b'_{ij} of (92) and (97), with shear taken into account, are given in Tables 3a, 3b and 3c. Potential-energy matrices for bending alone, h_{ij} of (55) and (95), and for shear alone, $(c_{ij} + d_{ij})/4h^2$ of (82) and (97), are given in Tables 4a and 4b.

The resulting values of the natural frequencies for the aforementioned six conditions are given in Table 5. The effects of shear are seen to be important, especially for the higher modes. Frequencies obtained by means of the added-mass matrix are somewhat lower than those obtained by means of strip theory, the percentage difference decreasing as the order of the mode increases, contrary to the effect of shear.

An estimate of the errors in the values of the natural frequencies due to the finite order of the matrix may be derived by applying Fig. 2 of Ref. [1]. The magnitudes of the errors, shown in Table 6, indicate that the results for the first six modes should be reasonably accurate.

TABLE 1

ADDED MASS MATRIX FOR BODY OF LENGTH L = 66.37'

0.003889	0.000279	0.000200	0.000095	0.000059	0.000037	0.000024
0.000279	0.013262	0.001271	0.000891	0.000418	0.000240	0.000141
0.000200	0.001271	0.022271	0.002244	0.001847	0.000793	0.000432
0.000095	0.000891	0.002244	0.030636	0.003187	0.002735	0.001126
0.000059	0.000418	0.001847	0.003187	0.039907	0.003719	0.003509
0.000037	0.000240	0.000793	0.002735	0.003719	0.045970	0.004192
0.000024	0.000141	0.000432	0.001126	0.003509	0.004192	0.048458
0.000016	0.000088	0.000250	0.000599	0.001449	0.003970	0.003847
0.000011	0.000059	0.000160	0.000359	0.000795	0.001690	0.004273
0.000008	0.000041	0.000106	0.000227	0.000472	0.000921	0.001777
0.000006	0.000029	0.000073	0.000150	0.000297	0.000545	0.000966
0.000004	0.000021	0.000052	0.000104	0.000197	0.000344	0.000574
0.000003	0.000016	0.000038	0.000074	0.000137	0.000229	0.000363
0.000003	0.000012	0.000029	0.000055	0.000098	0.000159	0.000241
0.000002	0.000009	0.000022	0.000040	0.000070	0.000110	0.000162
0.000002	0.000007	0.000016	0.000029	0.000050	0.000077	0.000110
0.000001	0.000005	0.000011	0.000020	0.000033	0.000050	0.000070
0.000001	0.000003	0.000007	0.000012	0.000020	0.000029	0.000040
0.000000	0.000001	0.000003	0.000005	0.000008	0.000011	0.000015

0.000016	0.000011	0.000008	0.000006	0.000004	0.000003	0.000003
0.000088	0.000059	0.000041	0.000029	0.000021	0.000016	0.000012
0.000250	0.000160	0.000106	0.000073	0.000052	0.000038	0.000029
0.000599	0.000359	0.000227	0.000150	0.000104	0.000074	0.000055
0.001449	0.000795	C.000472	0.000297	0.000197	0.000137	0.000098
0.003970	0.001690	0.000921	0.000545	0.000344	0.000229	0.000159
0.003847	0.004273	0.001777	0.000966	0.000574	0.000363	0.000241
0.048983	0.003961	0.004338	0.001789	0.000974	0.000579	0.000366
0.003961	0.050108	0.003938	0.004434	0.001829	0.000998	0.000593
0.004338	0.003938	0.050563	0.003947	0.004468	0.001845	0.001006
0.001789	0.004434	0.003947	0.050561	0.003945	0.004466	0.001844
0.000974	0.001829	0.004468	0.003945	0.050559	0.003942	0.004462
0.000579	0.000998	0.001845	0.004466	0.003942	0.050550	0.003958
0.000366	0.000593	0.001006	0.001844	0.004462	0.003958	0.050518
0.000236	0.000364	0.000580	0.000975	0.001792	0.004328	0.004046
0.000155	0.000229	0.000348	0.000550	0.000927	0.001710	0.004143
0.000097	0.000138	0.000202	0.000304	0.000483	0.000816	0.001523
C.000054	0.000075	C.000105	0.000153	0.000231	0.000367	0.000629
0.000020	0.000027	0.000037	0.000052	0.000075	0.000114	0.000184

Table 1 (Continued)

0.000002	0.000002	0.000001	0.000001	0.000000
0.000009	0.000007	0.000005	0.000003	0.000001
0.000022	0.000016	0.000011	0.000007	0.000003
0.000040	0.000029	0.000020	0.000012	0.000005
0.000070	0.000050	0.000033	0.000020	0.000008
0.000110	0.000077	0.000050	0.000029	0.000011
0.000162	0.000110	0.000070	0.000040	0.000015
0.000236	0.000155	0.000097	0.000054	0.000020
0.000364	0.000229	0.000138	0.000075	0.000027
0.000580	0.000348	0.000202	0.000105	0.000037
0.000975	0.000550	0.000304	0.000153	0.000052
0.001792	0.000927	0.000483	0.000231	0.000075
0.004328	0.001710	0.000816	0.000367	0.000114
0.004046	0.004143	0.001523	0.000629	0.000184
0.049059	0.003910	0.003603	0.001157	0.000311
0.003910	0.046634	0.003954	0.002769	0.000573
0.003603	0.003954	0.040799	0.003339	0.001386
0.001157	0.002769	0.003339	0.030943	0.001821
0.000311	0.000573	0.001386	0.001821	0.015311

Table 2a

KINETIC ENERGY MATRIX FOR BENDING -VIBRATIONS IN AIR -

0.077024	-0.054160	0.013540	0.0	0.0	0.0
-0.054160	0.057799	-0.010832	-0.0	-0.0	-0.0
0.013540	-0.010832	0.027002	0.0	0.0	0.0
0.0	-0.0	0.0	0.025147	0.0	0.0
0.0	-0.0	0.0	0.0	0.024559	0.0
0.0	-0.0	0.0	0.0	0.0	0.025471
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0

0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.027882	0.0	0.0	0.0	0.0	0.0
0.0	0.035471	0.0	0.0	0.0	0.0
0.0	0.0	0.039059	0.0	0.0	0.0
0.0	0.0	0.0	0.035176	0.0	0.0
0.0	0.0	0.0	0.0	0.033176	0.0
0.0	0.0	0.0	0.0	0.0	0.027529
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0

Table 2a (Continued)

0.0	0.0	0.0	-0.0	0.0
-0.0	-0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.025147	0.0	0.0	-0.0	0.0
0.0	0.031265	0.0	-0.0	0.0
0.0	0.0	0.024522	-0.005029	0.006286
-0.0	-0.0	-0.005029	0.034763	-0.025145
0.0	0.0	0.006286	-0.025145	0.041784

Table 2b

KINETIC ENERGY MATRIX FOR BENDING -VIBRATIONS IN WATER- STRIP THEORY

0.113516	-0.076755	0.019189	0.0	0.0	0.0
-0.076755	0.091169	-0.015351	-0.0	-0.0	-0.0
0.019189	-0.015351	0.050612	0.0	0.0	0.0
0.0	-0.0	0.0	0.055584	0.0	0.0
0.0	-0.0	0.0	0.0	0.060325	0.0
0.0	-0.0	0.0	0.0	0.0	0.063200
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0

0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.066028	0.0	0.0	0.0	0.0	0.0
0.0	0.074700	0.0	0.0	0.0	0.0
0.0	0.0	0.078640	0.0	0.0	0.0
0.0	0.0	0.0	0.074757	0.0	0.0
0.0	0.0	0.0	0.0	0.072757	0.0
0.0	0.0	0.0	0.0	0.0	0.067110
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0

Table 2b (Continued)

0.0	0.0	0.0	-0.0	0.0
-0.0	-0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.064728	0.0	0.0	-0.0	0.0
0.0	0.069551	0.0	-0.0	0.0
0.0	0.0	0.063828	-0.017684	0.022105
-0.0	-0.0	-0.017684	0.116681	-0.088420
0.0	0.0	0.022105	-0.088420	0.144034

Table 2c

KINETIC ENERGY MATRIX FOR BENDING -VIBRATIONS IN WATER- ADDED MASS MATRIX

0.115987	-0.072392	0.019669	0.000565	0.000332	0.000201
-0.072392	0.094826	-0.012567	0.001729	0.000719	0.000384
0.019669	-0.012567	0.058705	0.003216	0.002753	0.001138
0.000565	0.001729	0.003216	0.065054	0.003719	0.003509
0.000332	0.000719	0.002753	0.003719	0.070529	0.004192
0.000201	0.000384	0.001138	0.003509	0.004192	0.073929
0.000128	0.000218	0.000607	0.001449	0.003970	0.003847
0.000086	0.000138	0.000364	0.000795	0.001690	0.004273
0.000061	0.000090	0.000231	0.000472	0.000921	0.001777
0.000044	0.000061	0.000153	0.000297	0.000545	0.000966
0.000031	0.000044	0.000106	0.000197	0.000344	0.000574
0.000023	0.000032	0.000075	0.000137	0.000229	0.000363
0.000019	0.000023	0.000056	0.000098	0.000159	0.000241
0.000014	0.000018	0.000041	0.000070	0.000110	0.000162
0.000012	0.000012	0.000032	0.000054	0.000082	0.000117
0.000005	0.000003	0.000010	0.000017	0.000028	0.000040
0.000008	0.000012	0.000025	0.000040	0.000056	0.000077

0.000128	0.000086	0.000061	0.000044	0.000031	0.000023
0.000218	0.000138	0.000090	0.000061	0.000044	0.000032
0.000607	0.000364	0.000231	0.000153	0.000106	0.000075
0.001449	0.000795	0.000472	0.000297	0.000197	0.000137
0.003970	0.001690	0.000921	0.000545	0.000344	0.000229
0.003847	0.004273	0.001777	0.000966	0.000574	0.000363
0.076865	0.003961	0.004338	0.001789	0.000974	0.000579
0.003961	0.085579	0.003938	0.004434	0.001829	0.000998
0.004338	0.003938	0.089622	0.003947	0.004468	0.001845
0.001789	0.004434	0.003947	0.085737	0.003945	0.004466
0.000974	0.001829	0.004468	0.003945	0.083735	0.003942
0.000579	0.000998	0.001845	0.004466	0.003942	0.078079
0.000366	0.000593	0.001006	0.001844	0.004462	0.003958
0.000236	0.000364	0.000580	0.000975	0.001792	0.004328
0.000165	0.000242	0.000366	0.000576	0.000964	0.001767
0.000057	0.000084	0.000128	0.000200	0.000333	0.000588
0.000104	0.000142	0.000197	0.000283	0.000418	0.000652

Table 2c (Continued)

0.000019	0.000014	0.000012	0.000005	0.000008
0.000023	0.000018	0.000012	0.000003	0.000012
0.000056	0.000041	0.000032	0.000010	0.000025
0.000098	0.000070	0.000054	0.000017	0.000040
0.000159	0.000110	0.000082	0.000028	0.000056
0.000241	0.000162	0.000117	0.000040	0.000077
0.000366	0.000236	0.000165	0.000057	0.000104
0.000593	0.000364	0.000242	0.000084	0.000142
0.001006	0.000580	0.000366	0.000128	0.000197
0.001844	0.000975	0.000576	0.000200	0.000283
0.004462	0.001792	0.000964	0.000333	0.000418
0.003958	0.004328	0.001767	0.000588	0.000652
0.075665	0.004046	0.004235	0.001155	0.001089
0.004046	0.080324	0.004065	0.002981	0.001934
0.004235	0.004065	0.075557	-0.016839	0.030537
0.001155	0.002981	-0.016839	0.135420	-0.098538
0.001089	0.001934	0.030537	-0.098538	0.177526

Table 3a

KINETIC ENERGY MATRIX FOR BENDING AND SHEAR -VIBRATICNS IN AIR-

0.161792	-0.297698	0.118035	0.0	0.0	0.0
-0.297698	0.861690	-0.695481	0.168809	-0.0	-0.0
0.118035	-0.695481	1.357980	-1.035590	0.275362	0.0
0.0	0.168809	-1.035590	1.796350	-1.034380	0.168097
0.0	-0.0	0.275362	-1.034380	1.345930	-0.688176
0.0	-0.0	0.0	0.168097	-0.688176	0.967517
0.0	-0.0	0.0	0.0	0.116614	-0.516660
0.0	-0.0	0.0	0.0	0.0	0.089489
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0

0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.116614	0.0	0.0	0.0	0.0	0.0
-0.516660	0.089489	0.0	0.0	0.0	0.0
1.060860	-1.058920	0.358601	0.0	0.0	0.0
-1.058920	2.293150	-1.500360	0.282650	0.0	0.0
0.358601	-1.500360	2.289840	-1.462530	0.341297	0.0
0.0	0.282650	-1.462530	2.443180	-1.539290	0.321892
0.0	0.0	0.341297	-1.539290	2.342500	-1.367080
0.0	0.0	0.0	0.321892	-1.367080	2.002040
0.0	0.0	0.0	0.0	0.267101	-1.180250
0.0	0.0	0.0	0.0	0.0	0.241209
0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0

Table 3a (Continued)

0.0	0.0	0.0	-0.0	0.0
-0.0	-0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.267101	0.0	0.0	-0.0	0.0
-1.180250	0.241209	0.0	-0.0	0.0
1.782060	-1.068880	0.213446	-0.0	0.0
-1.068880	1.515580	-0.671856	0.062561	0.0
0.213446	-0.671856	0.682135	-0.274649	0.042088
-0.0	0.062561	-0.274649	0.392719	-0.120458
0.0	0.0	0.042088	-0.120458	0.072069

Table 3b

KINETIC ENERGY MATRIX FOR BENDING AND SHEAR -VIBRATIONS IN WATER-STRIP THEORY

0.287479	-0.570099	0.230329	0.0	0.0	0.0
-0.570099	1.693710	-1.356370	0.325013	-0.0	-0.0
0.230329	-1.356370	2.700820	-2.174550	0.608649	0.0
0.0	0.325013	-2.174550	3.963830	-2.384860	0.412902
0.0	-0.0	0.608649	-2.384860	3.241230	-1.697560
0.0	-0.0	0.0	0.412902	-1.697560	2.386340
0.0	-0.0	0.0	0.0	0.289350	-1.256060
0.0	-0.0	0.0	0.0	0.0	0.211922
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	-0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0

0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.289350	0.0	0.0	0.0	0.0	0.0
-1.256060	0.211922	0.0	0.0	0.0	0.0
2.431430	-2.290040	0.755195	0.0	0.0	0.0
-2.290040	4.826540	-3.097600	0.569077	0.0	0.0
0.755195	-3.097600	4.681660	-3.033230	0.725333	0.0
0.0	0.569077	-3.033230	5.182540	-3.322010	0.705928
0.0	0.0	0.725333	-3.322010	5.179460	-3.149800
0.0	0.0	0.0	0.705928	-3.149800	4.834620
0.0	0.0	0.0	0.0	0.651138	-2.953520
0.0	0.0	0.0	0.0	0.0	0.620869
0.0	0.0	0.0	0.0	0.0	0.0
-0.0	-0.0	-0.0	-0.0	-0.0	-0.0
0.0	0.0	0.0	0.0	0.0	0.0

Table 3b (Continued)

0.0	0.0	0.0	-0.0	0.0
-0.0	-0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.0	0.0	0.0	-0.0	0.0
0.651138	0.0	0.0	-0.0	0.0
-2.953520	0.620869	0.0	-0.0	0.0
4.476040	-2.573580	0.474824	-0.0	0.0
-2.573580	3.476370	-1.548290	0.159749	0.0
0.474824	-1.548290	1.697220	-0.768780	0.134151
-0.0	0.159749	-0.768780	1.207730	-0.386885
0.0	0.0	0.134151	-0.386885	0.238223

Table 3c

KINETIC ENERGY MATRIX FOR BENDING AND SHEAR -VIBRATIONS IN WATER-

ADDED MASS MATRIX

0.332380	-0.684312	0.265188	-0.014494	0.005170	0.000626
-0.684312	2.035890	-1.546990	0.372339	-0.023643	0.004447
0.265188	-1.546990	3.011480	-2.395640	0.691538	-0.034175
-0.014494	0.372339	-2.395640	4.401430	-2.625710	0.496703
0.005170	-0.023643	0.691538	-2.625710	3.574890	-1.873830
0.000626	0.004447	-0.034175	0.496703	-1.873830	2.638650
0.000749	0.000126	0.007044	-0.033818	0.365056	-1.417660
-0.001170	0.000683	-0.001956	0.005147	-0.031133	0.312331
0.000155	0.000013	0.000631	0.000186	0.005263	-0.041669
-0.000207	0.000196	-0.000193	0.000592	0.000675	0.004341
-0.000083	0.000079	0.000004	0.000210	0.000213	0.000454
-0.000070	0.000081	-0.000070	0.000212	0.000136	0.000202
-0.000075	0.000070	-0.000037	0.000192	0.000192	0.000193
-0.000058	0.000070	-0.000037	0.000128	0.000071	0.000113
0.000028	-0.000029	0.000019	-0.000049	-0.000038	-0.000062
-0.000059	0.000063	-0.000032	0.000128	0.000104	0.000121
0.000019	-0.000018	0.000015	-0.000041	-0.000047	-0.000051

0.000749	-0.001170	0.000155	-0.000207	-0.000083	-0.000070
0.000126	0.000683	0.000013	0.000196	0.000079	0.000081
0.007044	-0.001956	0.000631	-0.000193	0.000004	-0.000070
-0.033818	0.005147	0.000186	0.000592	0.000210	0.000212
0.365056	-0.031133	0.005263	0.000675	0.000213	0.000136
-1.417660	0.312331	-0.041669	0.004341	0.000454	0.000202
2.685410	-2.539530	0.884042	-0.059129	0.010201	0.000905
-2.539530	5.311410	-3.407670	0.728418	-0.064970	0.007561
0.884042	-3.407670	5.116390	-3.354490	0.889101	-0.067065
-0.059129	0.728418	-3.354490	5.686110	-3.672720	0.879958
0.010201	-0.064970	0.889101	-3.672720	5.697670	-3.505990
0.000905	0.007561	-0.067065	0.879958	-3.505990	5.350680
-0.000065	0.001205	0.009128	-0.070171	0.823493	-3.296360
0.000027	0.000455	0.001038	0.009307	-0.065563	0.767741
0.000281	-0.000344	0.000520	0.000620	0.007844	-0.048254
-0.000224	0.000549	-0.000073	0.000410	0.000571	0.004082
0.000132	-0.000244	0.000106	-0.000087	0.000081	0.000566

Table 3c (Continued)

-0.000075	-0.000058	0.000028	-0.000059	0.000019
0.000070	0.000070	-0.000029	0.000063	-0.000018
-0.000037	-0.000037	0.000019	-0.000032	0.000015
0.000192	0.000128	-0.000049	0.000128	-0.000041
0.000192	0.000071	-0.000038	0.000104	-0.000047
0.000193	0.000113	-0.000062	0.000121	-0.000051
-0.000065	0.000027	0.000281	-0.000224	0.000132
0.001205	0.000455	-0.000344	0.000549	-0.000244
0.009128	0.001038	0.000520	-0.000073	0.000106
-0.070171	0.009307	0.000620	0.000410	-0.000087
0.823493	-0.065563	0.007844	0.000571	0.000081
-3.296360	0.767741	-0.048254	0.004082	0.000566
4.963020	-2.847690	0.570086	-0.027552	0.002175
-2.847690	3.833270	-1.716750	0.216583	-0.016778
0.570086	-1.716750	1.873740	-0.850485	0.148122
-0.027552	0.216583	-0.850485	1.368840	-0.432263
0.002175	-0.016778	0.148122	-0.432263	0.269403

Table 4a

POTENTIAL ENERGY MATRIX FOR BENDING

3507100.	-7014200.	3507100.	0.	0.	0.
-7014200.	19382400.	-17722200.	5354000.	0.	0.
3507100.	-17722200.	32805500.	-26472900.	7882450.	0.
0.	5354000.	-26472900.	46007900.	-34013000.	9124060.
0.	0.	7882450.	-34013000.	52238400.	-33967500.
0.	0.	0.	9124060.	-33967500.	47704200.
0.	0.	0.	0.	7859690.	-30002200.
0.	0.	0.	0.	0.	7141400.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.

0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
7859690.	0.	0.	0.	0.	0.
-30002200.	7141400.	0.	0.	0.	0.
45560500.	-32553300.	9135240.	0.	0.	0.
-32553300.	54776900.	-40459500.	11094500.	0.	0.
9135240.	-40459500.	65073900.	-45310400.	11560700.	0.
0.	11094500.	-45310400.	68897900.	-46242800.	11560700.
0.	0.	11560700.	-46242800.	69364100.	-46242800.
0.	0.	0.	11560700.	-46242800.	69298100.
0.	0.	0.	0.	11560700.	-46110600.
0.	0.	0.	0.	0.	11494600.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.

Table 4a (Continued)

0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
11560700.	0.	0.	0.	0.
-46110600.	11494600.	0.	0.	0.
67236600.	-42384100.	9697410.	0.	0.
-42384100.	56463800.	-31753800.	6179510.	0.
9697410.	-31753800.	39184600.	-21897400.	4769190.
0.	6179510.	-21897400.	25256300.	-9538380.
0.	0.	4769190.	-9538380.	4769190.

Table 4b

POTENTIAL ENERGY MATRIX FOR SHEAR

5008470.	-10956600.	3359530.	6116930.	-3528310.	0.
-10956600.	32192900.	-24459400.	-10354500.	20098500.	-6520930.
3359530.	-24459400.	44807800.	-16633700.	-26083700.	24977200.
6116930.	-10354500.	-16633700.	44067500.	-15465300.	-23870600.
-3528310.	20098500.	-26083700.	-15465300.	49627900.	-19438900.
0.	-6520930.	24977200.	-23870600.	-19438900.	50447100.
0.	0.	-5967650.	20344000.	-16817400.	-26458400.
0.	0.	0.	-4204350.	14805700.	-12794000.
0.	0.	0.	0.	-3198510.	20920100.
0.	0.	0.	0.	0.	-7261550.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.

0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.
-5967650.	0.	0.	0.	0.	0.
20344000.	-4204350.	0.	0.	0.	0.
-16817400.	14805700.	-3198510.	0.	0.	0.
-26458400.	-12794000.	20920100.	-7261550.	0.	0.
59292500.	-23331200.	-29046200.	29445600.	-7461250.	0.
-23331200.	55845900.	-24402200.	-29845000.	32927600.	-9002570.
-29046200.	-24402200.	77257600.	-32528200.	-36010300.	36010300.
29445600.	-29845000.	-32528200.	82119400.	-32927600.	-36010300.
-7461250.	32927600.	-36010300.	-32927600.	88381800.	-35805100.
0.	-9002570.	36010300.	-36010300.	-35805100.	86947300.
0.	0.	-9002570.	36010300.	-36010300.	-30674100.
0.	0.	0.	-9002570.	35805100.	-35599900.
0.	0.	0.	0.	-8899980.	30468900.
0.	0.	0.	0.	0.	-6334470.
0.	0.	0.	0.	0.	0.

Table 4b (Continued)

0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
-9002570.	0.	0.	0.	0.
36010300.	-9002570.	0.	0.	0.
-36010300.	35805100.	-8899980.	0.	0.
-30674100.	-35599900.	30468900.	-6334470.	0.
72884100.	-23071800.	-25337900.	17735600.	-2533350.
-23071800.	56386300.	-21926000.	-6832530.	4241480.
-25337900.	-21926000.	45039900.	-21894800.	2549890.
17735600.	-6832530.	-21894800.	25017000.	-7690820.
-2533350.	4241480.	2549890.	-7690820.	3432800.

TABLE 5

NATURAL FREQUENCIES, C.P.S.

	Bending			Bending and Shear		
MODE	In Air	Strip Theory	Added-Mass Matrix	In Air	Strip Theory	Added-Mass Matrix
1	20.8	14.0	12.3	18.3	12.3	10.9
2	51.9	34.1	30.5	41.4	27.8	25.0
3	86.7	58.4	53.9	61.4	41.8	38.5
4	132.5	87.7	81.7	80.7	54.6	51.4
5	195.8	128.2	120.1	88.6	59.5	55.9
6	268.0	174.0	164.4	91.0	61.4	57.8
7	344.1	225.1	214.6	102.6	70.1	66.7
8	431.0	282.8	271.4	107.9	73.7	69.8
9	523.4	343.5	331.0	122.8	80.9	76.9
10	616.4	407.0	392.5	127.8	84.9	81.0
11	704.5	466.1	447.8	139.7	88.9	85.2
12	782.9	521.5	499.2	144.8	95.0	91.3
13	884.1	592.7	564.8	152.7	96.3	93.2
14	938.8	615.9	582.3	166.5	107.8	103.6
15	1044.8	679.8	640.4	176.3	109.9	106.2

TABLE 6

ERRORS IN NATURAL FREQUENCIES

Mode	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Error(%)	0.1	0.4	0.9	1.6	2.5	3.7	5.0	6.4	8.1	10.0	12	14	17	19	22

7. Acknowledgements

The computer programs given in the Appendix were prepared by Matilde Macagno, to whom the author here expresses his appreciation for successfully completing so complex a task. Discussions with Professor Kwan Rim of the University of Iowa Mechanics and Hydraulics Department were useful in the formulation of the expressions for potential energy.

8. References

[1] L. Landweber, "Vibration of a Flexible Cylinder in a Fluid," *Journal of Ship Research,* Vol. 11, No. 3, September 1967.

[2] S. Timoshenko, *Vibration Problems in Engineering,* D. Van Nostrand Company, Inc., New York, 1937.

APPENDIX

PROGRAM 1

PROGRAM FOR ADDED MASS-MATRIX

```
      REAL*8 X(20),F(20),FP(20),BE(19),C(19,19),W(19,19),CA(19,19),DX
     1(20),EL(19),CK(19),ME(19),LE(19),EM(19),CK2(19),D(19,19)
      H = 3.4D0
      DX(1) = 0.C77896D0
      DO 2 I = 1,19
    2 DX(I+1) = 0.102456D0
      DX(20) = 0.C77896D0
  100 READ(5,1) X,F,FP
    1 FORMAT(8F10.8)
      WRITE(6,4)(X(I),F(I),FP(I),I=1,19)
    4 FORMAT(45H          X(I)           F(I)            FP(I)    /(3F15.8))
      DO 10 J = 1,19
      G =DX(J+1) +DX(J)
      FF = FP(J) * FP(J)
      BE(J) = F(J) - FF
      D1 = SQRT(DX(J+1)*DX(J+1)+2.0D0* FP(J)) + F(J))
      D2 = SQRT(DX(J)*DX(J)-2.0D0* FP(J)) + F(J))
      D13 = D1**3
      D23 = D2**3
      F1 =((DX(J+1)+FP(J))/D1 +(DX(J)-FP(J))/ D2)/BE(J)
      F2 =(DX(J+1) /D13)+(DX(J)/ D23)
      F3 = F1*(F(J) - 2.0D0 * FF)
      F4 = 2.0D0 * FP(J) * ((1.0D0/D1) - (1.0D0/D2))
      EL(J) = F3 - F4
      CK(J) = F1 + F2
      H1 = DX(J+1) / (DX(J) * G )
      H1 = DX(J) / (DX(J+1) * G)
      H3 =(DX(J+1) -DX(J)) /(DX(J+1) *DX(J))
      CK2(J)= -(DX(J+1)*DX(J+1)/D13 - DX(J)*DX(J)/D23)
      U1 = DX(J+1) + FP(J) + D1
      U2 = D2 + FP(J) - DX(J)
      E3 = ALOG(U1/U2)
      E21 = 2.0D0 * F(J) * (F(J) - 2.0D0 * FF)
      E22 = FP(J)* (3.0D0* F(J) -4.0D0* FF)
      E2 = (E21 + E22 *DX(J+1))/(2.0D0*BE(J) * D1)
      E4 = (E21 - E22 *DX(J)) / (2.0D0*BE(J) * D2)
      EM(J) = FP(J) * E3 - E2 + E4
      DO  9 I = 1,19
      T1 = X(I) - X(J)
      T2 = T1 * T1
      DN = T2 + F(I)
      W(I,J) = 0.5D0*F(I) / (DN**1.5)
      C(I,J) = 0.5D0*(3.0D0*T1*FP(I)+T2-2.0D0*F(I))/(DN**2.5)
      K = J - 1
```

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```
      L = J + 1
      IF(I.EQ.K) GO TO 7
      IF(I.EQ.L) GO TO 8
      U =DX(I) + DX(I+1)
      W(I,J) = W(I,J) * U
      C(I,J) = C(I,J) * U
      GO TO 9
    7 W(K,J) = W(K,J) * DX(K)- EM(J) * H1
      C(K,J) = C(K,J) *DX(K) - CK2(J) * H1
      GO TO 9
    8 W(L,J) = W(L,J) *DX(L+1) + EM(J) * H2
      C(L,J) = C(L,J) * DX(L+1) + CK2(J) * H2
    9 CONTINUE
      W(J,J) = EL(J) + EM(J) * H3
   10 C(J,J) = - CK(J)+ CK2(J) * H3
      CALL MINV(C,19,DE,LE,ME)
      DO 11 I = 1,19
      DO 11 J = 1,19
      CA(I,J) = 0.0
      DO 11 K = 1,19
   11 CA(I,J) =  (CA(I,J) + W(I,K) * C(K,J) *(DX(J) + DX(J+1)))
      PI = 1.57C81C7D0
      FO = 31.650400 / H
      DO 12 I = 1,19
      DO 12 J = I,19
      D(I,J) = -(CA(I,J) + CA(J,I)) * PI * FO
   12 D(J,I) = D(I,J)
      WRITE(6,13)((D(I,J),J=1,10),I=1,19)
   13 FORMAT(27H  ADDED MASS MATRIX A(I,J)  /(10F12.6))
      WRITE(6,14)((D(I,J),J=11,19),I=1,19)
   14 FORMAT(1H 9F12.6)
 1000 CONTINUE
      WRITE(7,20) ((D(I,J),J=1,7),I=1,19)
   20 FORMAT(7D11.6)
      WRITE(7,20) ((D(I,J),J=8,14),I=1,19)
      WRITE(7,21) ((D(I,J),J=15,19),I=1,19)
   21 FORMAT(5D11.6)
      CALL EXIT
      END
```

APPENDIX

```
      PROGRAM 2

C     COMPUTATION OF MATRICES OF KINETIC ENERGY FOR BENDING A(I,J) AND B(I,J)

      REAL*8 EM(19),AM(19),D(19,19),E(19,19),F(19,19)
      II = 1
      READ(5,1) EM,AM
    1 FORMAT(7D11.6)
      READ(5,1)((D(I,J),J= 1, 7),I= 1,19)
      READ(5,1)((D(I,J),J= 8,14),I= 1,19)
      READ(5,7)((D(I,J),J=15,19),I= 1,19)
    7 FORMAT(5D11.6)
      DO 3 I = 1,19
      DO 2 J = 1,19
    2 E(I,J) = 0.0D0
    3 E(I,I) = EM(I)
  100 DO 11 I = 1,19
      DO 11 J = 1,19
   11 F(I,J) = E(I,J)
      DO 12 J = 5,15
      F(2,J) = E(2,J) + 2.5D0 * E(1,J)
      F(3,J) = E(3,J) - 2.0D0 * E(1,J)
      F(4,J) = E(4,J) + 0.5D0 * E(1,J)
      F(J,18) = E(18,J) + 2.5D0 * E(19,J)
      F(J,17) = E(17,J) - 2.0D0 * E(19,J)
   12 F(J,16) = E(16,J) + 0.5D0 * E(19,J)
      F(2,2) = E(2,2) + 6.25D0 * E(1,1) + 5.0D0 * E(1,2)
      F(2,3) = E(2,3)- 5.0D0 * E(1,1) - 2.0*E(1,2) + 2.5D0* E(1,3)
      F(2,4)=E(2,4)+1.25D0*E(1,1)+0.5D0*E(1,2)+2.5D0*E(1,4)
      F(3,3)=E(3,3)+4.0D0*E(1,1)-4.0D0*E(1,3)
      F(3,4)=E(3,4)-E(1,1)+0.5D0*E(1,3)-2.0D0*E(1,4)
      F(4,4)=E(4,4)+0.25D0*E(1,1)+E(1,4)
      F(16,16)=E(16,16)+0.25D0*E(19,19)+E(19,16)
      F(16,17)=E(17,16)-E(19,19)+0.5D0*E(19,17)-2.0D0*E(19,16)
      F(17,17)=E(17,17)+4.0D0*E(19,19)-E(19,17)
      F(16,18)=E(18,16)+1.25D0*E(19,19)+0.5D0*E(19,18)+2.5D0*E(19,16)
      F(17,18)=E(18,17)-5.0D0*E(19,19)-2.0D0*E(19,18)+2.5D0*E(19,17)
      F(18,18)=E(18,18)+6.25D0*E(19,19)+5.0D0*E(19,18)
      F(2,18)=E(2,18)+2.5D0*E(1,18)+6.25D0*E(1,19)+2.5D0*E(2,19)
      F(2,17)=E(2,17)+2.5D0*E(1,17)- 5.0D0*E(1,19)- 2.0D0*E(2,19)
      F(2,16)=E(2,16)+2.5D0*E(1,16)+1.25D0*E(1,19)+0.5D0*E(2,19)
      F(3,18)=E(3,18)-2.0D0*E(1,18)-5.0D0*E(1,19)+2.5D0*E(3,19)
      F(3,17)=E(3,17)-2.0D0*E(1,17)+4.0D0*E(1,19)-2.0D0*F(3,19)
      F(3,16)=E(3,16)-2.0D0*E(1,16)-E(1,19)+0.5D0*E(2,19)
      F(4,18)=E(4,18)+0.5D0*E(1,18)+1.25D0*E(1,19)+2.5D0*E(4,19)
      F(4,17)=E(4,17)+0.5D0*E(1,17)-E(1,19)-2.0D0*F(4,19)
      F(4,16)=E(4,16)+0.5D0*E(1,16)+0.25D0*E(1,19)+0.5D0*E(4,19)
      DO  13 I = 1,18
```

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```
      J = I + 1
      DO  13 K = J,19
   13 F(K,I) = F(I,K)
      DO 14 I = 2,18
      I1 = I - 1
      DO 14 J = 2,18
      J1 = J - 1
   14 F(I1,J1) = F(I,J)
      WRITE(6,5)((F(I,J),J= 1, 9),I=1,17)
    5 FORMAT(1H 9D12.5)
      WRITE(6,8)((F(I,J),J=10,17),I=1,17)
    8 FORMAT(1H 8D12.5)
      WRITE(7,21)((F(I,J),J=1  ,  5),I=1,17)
   21 FORMAT(5D15.6)
      WRITE(7,21)((F(I,J),J= 6,10),I=1,17)
      WRITE(7,21)((F(I,J),J=11,15),I=1,17)
      WRITE(7,22)((F(I,J),J=16,17),I=1,17)
   22 FORMAT(2D15.6)
      II = II + 1
      IF(II.EQ.3) GO TO 26
      IF(II.GT.3) GO TO 30
      DO 25 I = 1,19
      DO 24 J = 1,19
   24 E(I,J) = 0.0D0
   25 E(I,I) = EM(I) + AM(I)
      GO TO 100
   26 DO 28 I = 1,19
      DO 27 J = 1,19
   27 E(I,J) = D(I,J)
   28 E(I,I) = D(I,I) + EM(I)
      GO TO 100
   30 CONTINUE
      CALL EXIT
      END
```

APPENDIX

PROGRAM 3

PROGRAM OF KINETIC ENERGY FOR BENDING AND SHEAR A'(I,J) AND B'(I,J)

```
      REAL*8 G(19),H(19),AL(19),BE(19),EM(19),AM(19),B(19,19),D(19,19),
     1E(19,19),F(19,19)
      II = 1
      READ(5,1) EM,AM
    1 FORMAT (7D11.6)
      READ(5,2) G,H
    2 FORMAT(7D10.0)
      READ(5,1)((D(I,J),J= 1, 7),I=1,19)
      READ(5,1)((D(I,J),J= 8,14),I=1,19)
      READ(5,7)((D(I,J),J=15,19),I=1,19)
    7 FORMAT(5D11.6)
      DO 102 I = 1,19
      BE(I) = 0.0D0
      AL(I) = 0.0D0
      DO 101 J = 1,19
  101 B(I,J) = 0.0D0
  102 B(I,I) = EM(I)
      DX2 = 11.56D0
      DO 105 I = 2,18
      AL(I) = - H(I) / (G(I) * DX2)
  105 BE(I) = 1.0D0 - 2.0D0 * AL(I)
      BE(1) = 1.0D0
      BE(19) = 1.0D0
  120 DO 108 I = 2,18
      IP = I + 1
      IM = I - 1
      E(1,I) = AL(2)*AL(IP)*B(2,IP)+BE(I)*B(1,I)+AL(2)*BE(I)*B(2,I)+AL(
     1IP)*B(1,IP)+AL(IM)*B(1,IM)+AL(2)*AL(IM)*B(2,IM)
      E(I,1) = E(1,I)
      E(I,19)=BE(I)*B(I,19)+AL(IM)*AL(18)*B(IM,18)+AL(IP)*B(IP,19)+AL(IM
     1)*B(IM,19)+AL(18)*BE(I)*B(I,18)+AL(IP)*AL(18)*B(IP,18)
      E(19,I) = E(I,19)
      DO 108 J = 2,18
      JP = J + 1
      JM = J - 1
      E(I,J) = AL(IP)*AL(JP)*B(IP,JP)+ BE(I)*BE(J)*B(I,J)+AL(IM)*AL(JM)*
     1B(IM,JM)+AL(IP)*BE(J)*B(IP,J)+AL(JP)*BE(I)*B(I,JP)+AL(IM)*BE(J)*
     2B(IM,J)+AL(JM)*BE(I)*B(I,JM)+AL(IP)*AL(JM)*B(IP,JM)+AL(IM)*
     3B(IM,JP)*AL(JP)
  108 F(I,J) = E(I,J)
      E(1,1) = AL(2)*AL(2)*B(2,2)+B(1,1)+2.0D0*AL(2)*B(1,2)
      E(19,19)=AL(18)*AL(18)*B(18,18)+B(19,19)+2.0D0*AL(18)*B(18,19)
      E(1,19)=B(1,19)+AL(2)*B(2,19)+AL(18)*B(1,18)+AL(2)*AL(18)*B(2,18)
      DO 12 J = 5,15
```

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```
      F(2,J) = E(2,J) + 2.5D0 * E(1,J)
      F(3,J) = E(3,J) - 2.0D0 * E(1,J)
      F(4,J) = E(4,J) + 0.5D0 * E(1,J)
      F(J,18) = E(18,J) + 2.5D0 * E(19,J)
      F(J,17) = E(17,J) - 2.0D0 * E(19,J)
   12 F(J,16) = E(16,J) + 0.5D0 * E(19,J)
      F(2,2) = E(2,2) + 6.25D0 * E(1,1) + 5.0D0 * E(1,2)
      F(2,3) = E(2,3)- 5.0D0 * E(1,1) - 2.0*E(1,2) + 2.5D0* E(1,3)
      F(2,4)=E(2,4)+1.25D0*E(1,1)+0.5D0*E(1,2)+2.5D0*E(1,4)
      F(3,3)=E(3,3)+4.0D0*E(1,1)-4.0D0*E(1,3)
      F(3,4)=E(3,4)-E(1,1)+0.5D0*E(1,3)-2.0D0*E(1,4)
      F(4,4)=E(4,4)+0.25D0*E(1,1)+E(1,4)
      F(16,16)=E(16,16)+0.25D0*E(19,19)+E(19,16)
      F(16,17)=E(17,16)-E(19,19)+0.5D0*E(19,17)-2.0D0*E(19,16)
      F(17,17)=E(17,17)+4.0D0*E(19,19)-E(19,17)
      F(16,18)=E(18,16)+1.25D0*E(19,19)+0.5D0*E(19,18)+2.5D0*E(19,16)
      F(17,18)=E(18,17)-5.0D0*E(19,19)-2.0D0*E(19,18)+2.5D0*E(19,17)
      F(18,18)=E(18,18)+6.25D0*E(19,19)+5.0D0*E(19,18)
      F(2,18)=E(2,18)+2.5D0*E(1,18)+6.25D0*E(1,19)+2.5D0*E(2,19)
      F(2,17)=E(2,17)+2.5D0*E(1,17)- 5.0D0*E(1,19)- 2.0D0*E(2,19)
      F(2,16)=E(2,16)+2.5D0*E(1,16)+1.25D0*E(1,19)+0.5D0*E(2,19)
      F(3,18)=E(3,18)-2.0D0*E(1,18)-5.0D0*E(1,19)+2.5D0*E(3,19)
      F(3,17)=E(3,17)-2.0D0*E(1,17)+4.0D0*E(1,19)-2.0D0*E(3,19)
      F(3,16)=E(3,16)-2.0D0*E(1,16)-E(1,19)+0.5D0*E(2,19)
      F(4,18)=E(4,18)+0.5D0*E(1,18)+1.25D0*E(1,19)+2.5D0*E(4,19)
      F(4,17)=E(4,17)+0.5D0*E(1,17)-E(1,19)-2.0D0*E(4,19)
      F(4,16)=E(4,16)+0.5D0*E(1,16)+0.25D0*E(1,19)+0.5D0*E(4,19)
      DO  13 I = 1,18
      J = I + 1
      DO  13 K = J,19
   13 F(K,I) = F(I,K)
      DO 14 I = 2,18
      I1 = I - 1
      DO 14 J = 2,18
      J1 = J - 1
   14 F(I1,J1) = F(I,J)
      WRITE(6,5)((F(I,J),J= 1, 9),I=1,17)
    5 FORMAT(1H 9D12.5)
      WRITE(6,8)((F(I,J),J=10,17),I=1,17)
    8 FORMAT(1H 8D12.5)
      WRITE(7,21)((F(I,J),J=1 , 5),I=1,17)
   21 FORMAT(5D15.6)
      WRITE(7,21)((F(I,J),J= 6,10),I=1,17)
      WRITE(7,21)((F(I,J),J=11,15),I=1,17)
      WRITE(7,22)((F(I,J),J=16,17),I=1,17)
   22 FORMAT(2D15.6)
      II = II + 1
      IF(II.EQ.3) GO TO 26
      IF(II.GT.3) GO TO 30
```

```
      DO 25 I = 1,19
      DO 24 J = 1,19
   24 B(I,J) = 0.0D0
   25 B(I,I) = EM(I) + AM(I)
      GO TO 120
   26 DO 28 I = 1,19
      DO 27 J = 1,19
   27 B(I,J) = D(I,J)
   28 B(I,I) = C(I,I) + EM(I)
      GO TO 120
   30 CONTINUE
      CALL EXIT
      END
```

APPENDIX

PROGRAM 4

COMPUTATION OF MATRICES OF POTENTIAL ENERGY FOR BENDING AND SHEAR

```
C       A(I,J) =MATRIX OF POTENTIAL ENERGY FOR BENDING
C       C(I,J) =MATRIX OF POTENTIAL ENERGY FOR SHEAR
        REAL*8 H(19),G(19),C(19,19),A(19,19),U
        II = 1
        READ(5,1)G,H
      1 FORMAT(7D10.0)
        DO 2 J = 1,19
        DO 2 I = 1,19
        C(I,J) = 0.0D0
      2 A(I,J) = 0.0D0
        DO 3 I = 2,18
        M = I + 1
        N = I - 1
        A(I,I) =(G(M) + 4.0D0*G(I) + G(N))
        U = - 2.0D0*(G(I) + G(M))
        A(I,M) = U
      3 A(N,M) = G(I)
        I = 2
        DO 102 J = 6,18
        I1 = I + 1
        I2 = I + 2
        I3 = I + 3
        I4 = I + 4
        C(I,J) = - H(I2)
        C(I1,J) =   2.0D0 * (H(I2) + H(I3))
        C(I2,J) = - 4.0D0 * H(I3)
        C(I3,J) = - 2.0D0 * (H(I2) + H(I4+1))
        C(I4,J) = H(I2) + 4.0D0 * (H(I3) + H(I4 + 1)) + H(I4 + 2)
        I = I + 1
    102 CONTINUE
        H(2) = 46717220.0D0
        H(18) = 22511850.0D0
        C(2,2) = H(2) + 0.25D0*H(3) + H(4)
        C(2,3) = - 2.0D0*H(2) - H(3) - 2.0D0*H(4)
        C(2,4) = H(2) + 1.25D0 * H(3)
        C(2,5) = - 0.5D0 * H(3) + 2.0D0 * H(4)
        C(3,3) = 4.0D0 * (H(2) + H(3) + H(4)) + H(5)
        C(3,4) = - ( 2.0D0 * (H(2) + H(5)) + 5.0 * H(3))
        C(3,5) = 2.0D0 * H(3) - 4.0D0 * H(4)
        C(4,4) = H(2) + 6.25D0 * H(3) + 4.0D0 * H(5) + H(6)
        C(4,5) = - (2.5D0 * H(3) + 2.0D0 * H(6))
        C(5,5) = H(3) + 4.0D0 * (H(4) + H(6)) + H(7)
        C(18,18) = H(18) + 0.25D0 * H(17) + H(16)
        C(17,18) =  -(2.0D0 * (H(18) + H(16)) + H(17))
```

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```
      C(16,18) = H(18) + 1.25D0 * H(17)
      C(15,18) = - 0.5D0 * H(17) + 2.0D0 * H(16)
      C(17,17) = 4.0D0 * (H(18) + H(17) + H(16)) + H(15)
      C(16,17) = - ( 2.0D0 * (H(18) + H(15)) + 5.0D0 * H(17))
      C(15,17) = 2.0D0 * H(17) - 4.0DC * H(16)
      C(16,16) = H(18) + 6.25D0 * H(17) + 4.0D0 * H(15) + H(14)
      C(15,16) = - ( 2.5D0 * H(17) + 2.0D0 * H(14))
      C(15,15) = H(17) + 4.0D0 * (H(16) + H(14)) + H(13)
      DO 103 I = 1,18
      J = I + 1
      DO 103 K = J,19
      A(K,I) = A(I,K)
  103 C(K,I) = C(I,K)
      DO 104 I = 2,18
      I1 = I - 1
      DO 104 J = 2,18
      J1 = J - 1
      A(I1,J1) = A(I,J)
  104 C(I1,J1) = C(I,J) / 46.24D0
      WRITE(6,5)((A(I,J),J= 1, 9),I=1,17)
    5 FORMAT(1H 9D12.5)
      WRITE(6,8)((A(I,J),J=10,17),I=1,17)
    8 FORMAT(1H 8D12.5)
      WRITE(6,5)((C(I,J),J= 1, 9),I=1,17)
      WRITE(6,8)((C(I,J),J=10,17),I=1,17)
      WRITE(7,21)((A(I,J),J=1 , 5),I=1,17)
   21 FORMAT(5D15.6)
      WRITE(7,21)((A(I,J),J= 6,10),I=1,17)
      WRITE(7,21)((A(I,J),J=11,15),I=1,17)
      WRITE(7,22)((A(I,J),J=16,17),I=1,17)
   22 FORMAT(2D15.6)
      WRITE(7,21)((C(I,J),J=1 , 5),I=1,17)
      WRITE(7,21)((C(I,J),J= 6,10),I=1,17)
      WRITE(7,21)((C(I,J),J=11,15),I=1,17)
      WRITE(7,22)((C(I,J),J=16,17),I=1,17)
      CALL EXIT
      END
```

APPENDIX

PROGRAM 5

```
FREQUENCIES OF VIBRATIONS

      REAL*8 A(17,17),B(17,17),C(17,17),D(17,17),E(17,17),F(17,17),G(17
     1,17),H(17,17),U(17,17),V(17,17),S(17,17),R(17,17),T(17,17),X(17,17
     2),AL(17),BL(17),CL(17),DL(17),EL(17),FL(17),W1(17),W2(17),W3(17),
     4W4(17),W5(17),W6(17),Z
      READ(5,1)((A(I,J),J= 1, 5),I=1,17)
    1 FORMAT(5D15.6)
      READ(5,1)((A(I,J),J= 6,10),I=1,17)
      READ(5,1)((A(I,J),J=11,15),I=1,17)
      READ(5,2)((A(I,J),J=16,17),I=1,17)
    2 FORMAT(2D15.6)
      READ(5,1)((B(I,J),J= 1, 5),I=1,17)
      READ(5,1)((B(I,J),J= 6,10),I=1,17)
      READ(5,1)((B(I,J),J=11,15),I=1,17)
      READ(5,2)((B(I,J),J=16,17),I=1,17)
      READ(5,1)((C(I,J),J= 1, 5),I=1,17)
      READ(5,1)((C(I,J),J= 6,10),I=1,17)
      READ(5,1)((C(I,J),J=11,15),I=1,17)
      READ(5,2)((C(I,J),J=16,17),I=1,17)
      READ(5,1)((D(I,J),J= 1, 5),I=1,17)
      READ(5,1)((D(I,J),J= 6,10),I=1,17)
      READ(5,1)((D(I,J),J=11,15),I=1,17)
      READ(5,2)((D(I,J),J=16,17),I=1,17)
      READ(5,1)((E(I,J),J= 1, 5),I=1,17)
      READ(5,1)((E(I,J),J= 6,10),I=1,17)
      READ(5,1)((E(I,J),J=11,15),I=1,17)
      READ(5,2)((E(I,J),J=16,17),I=1,17)
      READ(5,1)((F(I,J),J= 1, 5),I=1,17)
      READ(5,1)((F(I,J),J= 6,10),I=1,17)
      READ(5,1)((F(I,J),J=11,15),I=1,17)
      READ(5,2)((F(I,J),J=16,17),I=1,17)
      READ(5,1)((G(I,J),J= 1, 5),I=1,17)
      READ(5,1)((G(I,J),J= 6,10),I=1,17)
      READ(5,1)((G(I,J),J=11,15),I=1,17)
      READ(5,2)((G(I,J),J=16,17),I=1,17)
      READ(5,1)((H(I,J),J= 1, 5),I=1,17)
      READ(5,1)((H(I,J),J= 6,10),I=1,17)
      READ(5,1)((H(I,J),J=11,15),I=1,17)
      READ(5,2)((H(I,J),J=16,17),I=1,17)
      Z = 72.36D0
      DO 3 I = 1,17
      DO 3 J = 1,17
      S(I,J) = G(I,J) + H(I,J)
      U(I,J) = G(I,J)
      V(I,J) = G(I,J)
```

```
      R(I,J) = S(I,J)
    3 T(I,J) = S(I,J)
      CALL DNROOT(17,G,A,AL,X)
      CALL DNROOT(17,U,B,BL,X)
      CALL DNROOT(17,V,C,CL,X)
      CALL DNROOT(17,S,D,DL,X)
      CALL DNROOT(17,R,E,EL,X)
      CALL DNROOT(17,T,F,FL,X)
      WRITE(6,4)(I,AL(I),BL(I),CL(I),DL(I),EL(I),FL(I),I=1,17)
    4 FCRMAT(30H1EIGEN VALUES OF SIX PROBLEMS /(I5,6D18.6))
      DO 8 J = 1,15
      I = 16 - J
      W1(I) = DSQRT(AL(J)) / Z
      W2(I) = DSQRT(BL(J)) / Z
      W3(I) = DSQRT(CL(J)) / Z
      W4(I) = DSQRT(DL(J)) / Z
      W5(I) = DSQRT(EL(J)) / Z
    8 W6(I) = DSQRT(FL(J)) / Z
      WRITE(6,9)(I,W1(I),W2(I),W3(I),I=1,15)
    9 FORMAT(40H1 FREQUENCIES OF VIBRATICN FOR BENDING  ///
     160H          IN AIR            STRIP TH.            WITH A.M.MATRIX /
     2//(I4,3F15.5))
      WRITE(6,11)(I,W4(I),W5(I),W6(I),I=1,15)
   11 FORMAT(48H FREQUENCIES OF VIBRATICN FOR BENDING AND SHEAR ///
     160H          IN AIR            STRIP TH.            WITH A.M.MATRIX /
     2//(I4,3F15.5))
      CALL EXIT
      END
```

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DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Iowa Institute of Hydraulic Research	2a. REPORT SECURITY CLASSIFICATION Unclassified	
	2b. GROUP	
3. REPORT TITLE Natural Frequencies of a Body of Revolution Vibrating Transversely in a Fluid		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Institute Report on Sponsored Research		
5. AUTHOR(S) (First name, middle initial, last name) Louis Landweber		
6. REPORT DATE August 1968	7a. TOTAL NO. OF PAGES 56	7b. NO. OF REFS 2
8a. CONTRACT OR GRANT NO. Nonr-3271(01)(X) b. PROJECT NO. SF 113 1108 c. Task 1360 d.	9a. ORIGINATOR'S REPORT NUMBER(S) IIHR Report No. 111 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Ship Research and Development Center	

13. ABSTRACT

A previously developed procedure, in which the natural frequencies of a body vibrating in a fluid are given by the eigenvalues of the potential energy matrix of an elastic body with respect to the combined kinetic energy matrix of the body and fluid, is applied to a body of revolution. Integral relations between the velocity potential on the body and the vibration velocity, and the expression for the kinetic energy of the fluid as a surface integral are discretized to obtain a quadratic form for the fluid energy. Integrals for the kinetic and potential energies of the elastic body, taking both bending and shear into account, are also discretized. The method is illustrated by application to a form with assumed inertial and elastic characteristics.

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Ship vibration						
Potential flow						
Natural frequencies						

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