058:268 Turbulent flows
G. Constantinescu

Handout: Large Eddy Simulation I

Introduction to Subgrid-Scale (SGS) Models

SGS Stresses should depend on:

- Local large-scale field

or

- Past history of local fluid (via PDE)

Not all models have these properties

Importance of model in a calculation depends on energy in subgrid-scales

- Low Reynolds number: $E_{SGS}/E \approx$ 10-50%; results relatively insensitive to model, however results can be very sensitive to the numerics if artificial dissipation is present (e.g., convective terms are discretized using upwind schemes)
- High Reynolds number: $E_{SGS}/E \approx 1$; model more important

Requirements that a good SGS/SFS model must fulfill:

- represent interaction with small scales
- the most important feature of a SGS model is to provide adequate dissipation (by this we mean transport of energy from the resolved grid scales to the unresolved grid scales; the rate of dissipation ε in this context is the flux of energy through the inertial subrange)
- the dissipation rate must depend on the large scales of the flow rather than being imposed arbitrarily by the model. The SGS model must depend on the large-scale statistics and must be sufficiently flexible to adjust to changes in these statistics.
- especially in energy conserving codes (ideal for LES) the only way for t.k.e. to leave the resolved modes is by the dissipation provided by the SGS model.
- primary goal of an SGS model is to obtain correct statistics of the energy-containing scales of motion

All the above observation suggest the use of an eddy viscosity type SGS model

- Take idea from RANS modeling, introduce eddy viscosity ν_T :

$$\tau_{ij} = -\nu_T \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) = -2\nu_T \overline{S}_{ij}$$

Simplest model has **constant eddy viscosity:**

- Really DNS of lower Re number flow
- Spectrum not controlled
- Not successful in practice
- Better model due to Smagorinsky

Smagorinsky model

- Dimensionally eddy viscosity is $l^2 t^{-1}$
- Obvious choice: $\nu_T = Cql$

  - Turbulence length scale easy to define (unlike in RANS): largest size of unresolved scales is approximately Δ

  $l = \Delta$

  - Velocity scale not obvious (smallest resolved scales, their size is of the order of the variation of velocity over one grid element)

  - use $q = l \frac{\partial u}{\partial y} = lS$

  - Better choice for 3D flows:

  $S = \left(2S_{ij}S_{ij}\right)^{1/2}$

  Observation: In RANS pretty much the inverse story, velocity scale is obvious ($k^{1/2}$) while the turbulent length scale is tough to define and estimate

  - Combine previous expressions to obtain

  $\nu_T = C\Delta^2 S = C_S^2 \Delta^2 S$

- The above model is due to Smagorinsky (1963)
  - Designed for global weather modeling
  - Used for the first time in 1962-63 at NCAR.
  - Original grid contained 20*20*20 points.
  - Nowadays simulations using couple of million points are current!
- Central problem: **Need to specify value for parameter C**

More physical argument to derive Smagorinsky model:

- Let's look at the energy transfer from large scales

  - Due to nonlinear term in NS equations
- To obtain energy equation, take scalar product of NS equations with u_i

$$\frac{\partial}{\partial t}\frac{u_i u_i}{2} + u_j \frac{\partial}{\partial x_j}\frac{u_i u_i}{2} = -u_i \frac{\partial p}{\partial x_i} + \nu u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

- Second term describes the energy transfer between scales (recall previous discussion in wavenumber space), dimensionally vel_scale^3/length_scale
- Energy transfer = dissipation = $\varepsilon \approx Q^3 / L$

  - L=integral scale, Q=large scale velocity
- If largest unresolved eddies are inviscid (close to reality if Re very large): $\varepsilon \approx q^3 / \Delta$
- Based on dimensional analysis:

  $\nu_T \approx q\Delta$
- By equating the two expressions for ε:

  $q / Q = (\Delta / L)^{1/3}$

  $\nu_T \approx q\Delta \approx Q(\Delta / L)^{1/3}\Delta = Q\Delta^{4/3}L^{-1/3}$
- Further assume:

  $Q \approx L(S_{ij}S_{ij})^{1/2} = L|S|$
- Then:

$$\nu_T = C\Delta^{4/3}L^{2/3}|S|$$

- This expression is somewhat different than the classical Smagorinsky model (two length scales are present) but, supposing one can estimate L, this form of the model is a more accurate expression for the SGS eddy viscosity. This expression is also obtained analytically using more advanced theories of turbulence (EQDVM model).
- Integral scale L difficult to compute, but we can assume:

  L/Δ=ct ; in reality L/Δ=f(Re, etc.)
- So we recover usual form of model (the value of the constant changed)

$$\nu_T \approx C\Delta^2 S$$

- In reality C=C(L/Δ)=C(Re, etc.)
- May explain why variation of C needed to obtain accurate prediction of turbulent flows (this is going to be addressed later via dynamically calculating the model coefficient C)

Smagorinsky Model can be derived in several ways:

- Heuristically (two versions given above)
- Inertial range arguments (Lilly)
- Turbulence Theories (RNG)

Constant predicted by all methods (based on theory, decay isotropic turbulence)

$$C_S = \sqrt{C} \approx 0.2$$

Theories of turbulence suggest a <u>spectral eddy viscosity</u> (Chollet and Lesieur)

- Means a different eddy viscosity acting on each wave number k
- Analogy with ordinary viscosity
- Energy removed from wave number k is

$$\varepsilon(k) = 2\nu k^2 E(k)$$

- This suggest a spectral eddy viscosity of the form:

$$\nu_T(k) = T_>(k)/2k^2E(k) \qquad \text{as } \varepsilon(k) \sim T_>(k)$$

- $T_>(k)$=net energy transfer to small scales

It can be shown based on DNS calculations (isotropic turbulence) and a priory analysis that:

- Spectral eddy viscosity constant at small k
- Increases rapidly at large k (small scales which in a numerical simulation on a finite grid correspond to the cutoff wave number k_c)

  - Reason: main interaction is between smallest resolved scales and largest unresolved scales. Eddy viscosity is largest between $k_c/2$ and k_c.
- Findings can be used to construct models

Parameter estimation (based on Lilly's theory)

Assume high Reynolds Number

- Cutoff lies in inertial subrange (no prod, no diss, E=E(ε,k), k is kinetic energy)
- Energy spectrum corresponding to the velocity is:

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3}$$

- C_K=Kolmogorov constant ≈1.4 - 2.2
- In kinetic energy equation for resolved scales:

$$\varepsilon_{RS} = \nu_T |S|^2 = C_S{}^2 |\overline{S}|^3 \Delta^2$$

- Estimate the square of the strain rate:

$$|\overline{S}|^2 = 2\int_0^{k_c} k^2 E(k) dk \approx \frac{3}{2} C_K \varepsilon^{2/3} k_c{}^{4/3}$$

- Use $k_c = \pi / \Delta$, $\varepsilon_{RS} = \varepsilon$ and previous two expressions to obtain

$$C_S = \frac{1}{\pi}\left(\frac{2}{3C_K}\right)^{3/4}$$

With $C_K = 1.6 \Rightarrow C_S = 0.165$

- Other methods give almost same value

Classical Smagorinsky model

Performance

- Predicts many flows reasonable well
- But there are **problems**:

- Optimum parameter value varies with flow type

*Isotropic turbulence $C_S \approx 0.2$

*Shear flows (e.g. channel) $C_S \approx 0.065$

*Factor 10 difference in eddy viscosity!!

- Length scale uncertain with anisotropic filter

*Two possibilities are:

$(\Delta_1\Delta_2\Delta_3)^{1/3} \quad ; \quad (\Delta_1{}^2 + \Delta_2{}^2 + \Delta_3{}^2)^{1/2}$

- Needs modification to account for:

*Rotation, stratification C_S=F(Ri,….) Ri=Richardson number

*Near-wall region C_S=F(y^+); viscosity comes into play resulting in the need for further reduction of the model coefficient. Van Driest damping is usually used but the results are not very good.

Ways to improve the model:

- dynamically calculate the model coefficient. This is the **dynamic Smagorinsky model**
- introduce transport equations for relevant quantities. In particular, solving an equation for the **subgrid kinetic energy** allows a much better estimation of the velocity scale for the SGS fluctuations. These are **the one-equation SGS models**.
- Both types of models are going to be discussed in details later.

A totally different approach (not based on eddy viscosity SGS models):

Use small scales of LES itself, the smallest resolved scales are not very different from the largest unresolved scales. Use that information for model construction. This is the main idea behind the **scale similarity model** (Bardina et al.)

Scale similarity model

First model based on the small resolved scales:

- We already made the point that the **most important interactions involve interactions between the Largest subgrid scales and Smallest resolved scales**
- Need to define these scales

Define velocity fields:

- Unresolved scales $(D \le \Delta)$:

  $u_i' = u_i - \overline{u}_i$ (by definition)

- Largest subgrid scale part defined by filtering:

  $\overline{u_i'} = \overline{u}_i - \overline{\overline{u}}_i$

- Smallest resolved scales $(D > \Delta)$;defined by second filter on the resolved field (of larger width, generally)

  $\overline{u}_i - \overline{\overline{u}}_i$

  Last two expressions are identical !!!

- So we assume that these scales have similar structure (near grid cutoff). In other words, we simply assume that the SGS stresses for the full velocity field are the same as the ones corresponding to the resolved field $\overline{u}_i$

  $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j \sim \overline{\overline{u}_i \overline{u}_j} - \overline{\overline{u}}_i \overline{\overline{u}}_j$

These ideas are clearer by looking at the following analysis of the relation among the different velocity fields

Full Field

u_i

Filter

Resolved Field	**Unresolved Field**
$\overline{u}_i$	$u'_i = u_i - \overline{u}_i$

Filter Again

Largest	**Transfer** (border between resolved/unresolved)	**Smallest**
$\overline{\overline{u}}_i$	$\overline{u}'_i = \overline{\overline{u}}_i - \overline{u}_i$	$u''_i = u'_i - \overline{u}'_i$

Another relevant discussion is related to what is called **<u>a priori</u> testing of LES models** using a precalculated DNS database (no actual LES simulation is needed).

A priori testing

<u>Full DNS Field</u>

u_i

<u>Filter</u>

Resolved Field	Unresolved Field
$\overline{u}_i$	$u'_i = u_i - \overline{u}_i$

<u>Evaluation</u>

Model SGS stresses	Exact SGS stresses
$-2\nu_T\overline{S}_{ij}$ (for Smagorinsky)	$\overline{u_i u_j} - \overline{u}_i\overline{u}_j$ (from DNS)
$\overline{\overline{u}_i\overline{u}_j} - \overline{\overline{u}}_i\overline{\overline{u}}_j$ (scale similarity)	

Finally, compare results by correlation or scatter plot. This way one can draw conclusions about the performance of a particular LES model.

Other way to derive the scale similarity model

Later, we are going to derive the scale similarity model as a particular case of a more general class of models based on reconstruction of the total velocity field from the resolved one (defiltering). The main idea in these models is:

- Use definition of resolved velocity:

$$\overline{u}_i(x) = \int G(x, x')u_i(x')dx'$$

- Apply Taylor series to $\overline{u}_i(x)$:

$$u_i(x') \approx u_i(x) + (x' - x)\frac{du_i}{dx} + \ldots$$

- Keep only first term into definition of $u_i(x')$ to obtain same result:

$$\tau_{ij} = \overline{\overline{u}_i\overline{u}_j} - \overline{\overline{u}}_i\overline{\overline{u}}_j$$

Performance of scale-similarity model:

- Improves energy spectra (compared to Smagorinsky)
- Can account for the transfer of energy from Small resolved scales → large resolved scales (backscatter accounted in a physical way)
- Correlates well with exact stress (a priori analysis)
- Not dissipative (does not dissipate energy automatically as Smagorinsky model with constant coefficient does, e.g., in laminar region of a flow the eddy viscosity and turbulence dissipation predicted by Smagorinsky model will be different from zero and positive, which is obviously wrong)
- Inadequate as stand-alone SGS model (not very robust numerically as it does not introduce enough dissipation in some cases, needs to be combined with a purely dissipative model, e.g., Smagorinsky like; this is the main idea behind mixed models to be discussed later)
- Basis for other models